Philosophical Sanity, Mysteries of the Understanding, and Dialectical Logic

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ABSTRACT. Hegel's *Logic*, it is said, makes claims of a big order; claims which, as far as modern logic is concerned, cannot be upheld. Against this, the authors maintain that it is modern logic itself which has not come to grips with the very problems which lie at the bottom of Hegel's speculative philosophy and which show up in modern logic as paradoxes, incompleteness, and undecidability results. This paper is a plea for taking advantage of the failure of Frege's original conception of (higher order) logic for the development of a dialectical logic. It aims, in particular, at a younger generation of Hegel students, who are neither caught in the paradigm of logic as truth functional, nor reject wholesale deductive methods as inappropriate for the purpose of formulating a logic which aims at capturing its own factual content. The authors suggest that classical logic is to be given up in favour of a so-called 'substructural logic' which allows for unrestricted abstraction. Unrestricted abstraction, by way of its capacity to create all forms of self-reflexivity, is the source of an abundance of strange phenomena which lend themselves much better to Hegel's dialectic than to the dogmas of the understanding.

1. Hegel's Kantian Legacy

The point of Hegel's idea of a first philosophy that is at once a logic, metaphysics and ontology is to establish a logical foundation of thought-forms. In this he can be said to repeat Kant's question: how can subjective conditions of thought have objective validity.[1] Certainly, he sees Kant as having begun to turn metaphysics to logic. This turn however, and also its further development by Fichte, remained for Hegel seriously incomplete.

[1] Immanuel Kant, *Critique of Pure Reason*, trans. by Norman Kemp Smith (London: Macmillan, 1933), pp. 120–124.

> The critical philosophy did already turn metaphysics into logic, but for fear of the object it, like the later idealism ... gave the logical determinations an essentially subjective meaning (*Bedeutung*); thereby they remained at the same time afflicted with the object they had fled, and a thing-in-itself, an infinite impetus, remained with them as a beyond (*SW* v.4, p. 47; *SL*, p. 51).[2]

Hegel's idea then was to complete this turn by reconceiving Kant's thing-in-itself as an abstraction or extension of reason: as the Reasonable.

Unlike Kant, who in his *Prolegomena to any Future Metaphysics* declares that only Hume's doubts "can be of the smallest use"' in the "perfectly new" science of metaphysics he had established,[3] Hegel's idea of metaphysics is not confined to the epistemological concerns of modern philosophy but also draws in the concerns of ancient and medieval metaphysics with content and substance.

> The objective logic ... takes the place of the former *metaphysics* ... — If we show consideration for the last form of the development of this science, then firstly it is immediately the *ontology*, the place of which is taken by the objective logic, — the part of that metaphysics which was meant to investigate the nature of the *ens* in general ... But then the objective logic also comprises the remaining metaphysics in so far as this attempted to grasp with the pure forms of thought particular substrata, initially taken from figurate conception (*Vorstellung*), the soul, the world, God ... (*SW* v.4, pp. 64–65; *SL*, p. 63).

Logically dealt with, according to Hegel, these forms of thought are freed from their submergence in self-conscious intuition and its substrata in 'figurate conception' (*Vorstellung*), that is, conception that is reliant on the myths and metaphors of sensuous perception.[4] Pre-Kantian metaphysics

[2] All quotations from Hegel are our translations, based on the fourth edition of the *Jubiläumsausgabe of Hegels Sämtliche Werke* (Stuttgart-Bad Cannstatt: Friedrich Frommann Verlag, 1961–8) cited hereafter as '*SW*' v. followed by the initials of the English translations noted. Here: *Science of Logic*, trans. by A. V. Miller (Atlantic Highlands, NJ: Humanities Press International, 1969). Cited hereafter as '*SL*'.

[3] Immanuel Kant, *Prolegomena To Any Future Metaphysics That Will be Able to Come Forward As Science*, trans. by Paul Carus, rev. by James W. Ellington (Indianapolis: Hackett Publishing Company, 1977), p. 7.

[4] "The myth is always a presentation which uses sensuous mode, introduces sensuous pictures, which are suited for the presentation, not to the thought; it is an

(though not, in their speculative moments Plato and Aristotle) omitted to do this and consequently

> incurred the just reproach of having employed these forms *without critique*, without a preliminary investigation, if and how they were capable of being determinations of the thing-in-itself to use the Kantian expression or rather of the Reasonable. The objective logic is therefore the true critique of them a critique which ... considers them themselves in their specific content (*SW* v.4, p. 65; *SL*, p. 64).

Reconstruction of the object, within a philosophical context in which historically it has been placed over against the subject entails reconstruction of the idea of subjectivity. Hegel's logic contains a third part, the doctrine of the Notion which he terms 'subjective' in the sense of being concerned with the subject itself: not the human subject but the living being of reason. In terms then of ideas of the subject and the subjective as opposed to the object and the objective, Hegel's logic is subject-less. It is not a phenomenology of spirit or consciousness. It is, or rather claims to be, a demonstration of how, taking nothing from outside, the totality of all determinations of pure thought, is derivable. In this, the science of logic takes a circular path which leads back to Being. This starting point however, is now enriched. It has been discovered that it contains all that succeeds it within itself. It has been 'ensouled by the method' and *acts* now to constitute the beginning of and for a new science.

> By dint of the demonstrated nature of the method, the science presents itself as a *circle* coiled in itself, into the beginning of which, the simple ground, the mediation coils back the end; in the process this circle is a *circle of circles*; for every single link, as ensouled by the method is the reflection into itself, which, in returning into the beginning is at the same time the beginning of a

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impotence of the thought, which does not yet know for itself how to hold on to itself, get by with itself. The mythical presentation, as older, is presentation where the thought is not yet free: it is pollution of the thought by sensuous form; this cannot express what the thought wants. It is appeal, a way of attracting, to occupy oneself with the content. It is something pedagogical. The myth belongs to the pedagogy of the human race. When the notion has grown up, it is no longer in need of it." *SW* v.18, pp. 188–9; G. W. F. Hegel, *Lectures on the History of Philosophy* (hereafter cited as '*LHP*') v.2, trans. by E. S. Haldane and Frances H. Simon (Lincoln: University of Nebraska Press, 1995), p. 20.

> new link. Fragments in this chain are the individual sciences, each of which has a *Before* and an *After*, or, more strictly speaking, only *has* the Before and in its very own ending *shows* its *After* (*SW* v.5, p. 351; *SL*, p. 842).

This 'simple ground' is not, as in Kant, the transcendental unity of apperception or, as in Fichte the ego or 'I'. It is logical *abstraction* in the sense of passing from a predicate to its extension — a sense of abstraction commonly termed 'reification'. Outrageously, Hegel's logical project claims to find within and by means of speculative reason, a logic that is not just a canon of judgements but an '*organon* for the production of *objective* insights' (*SW* v.5, 23; *SL*, 590). That is to say Hegel's logic claims not only to be *truth preserving* but to be *truth generative*. It involves nothing less than an attempt to derive from within thought, not only the *validity* and *value* of its categories for 'objective truth' (pace Kant) but also their *substance* and *content*.

In the face of the modern transformation of logic into an essentially mathematical discipline, is it open to read Hegel's logic *as a logic*? If Hegel's logic cannot be read as a logic then either a methodologistic (neo-Kantian) or broadly hermeneutic interpretation must be the best interpretive bet. Yet thinking about thinking, is evidently self-referential. Some will say that we 'ought not' engage in so silly an enterprise. Such is Carnap's doctrine of the "confusion of spheres",[5] strongly modelled after Russell's simple theory of types. But there is still the possibility that the contradictions and paradoxes of self-reference have epistemological significance and it is this possibility that we want to open and leave open.

If Hegel's idea of his own philosophy as grounded in a speculative logic is dismissed, then he will be interpreted within a framework of thought for which classical logic continues to play its particular truth preserving role. This role is premised on truth definiteness, that is the validity of either-or reasoning as applied to the truth values true and false.[6] Far from regarding this form of reasoning as canonical, Hegel links it to the

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[5] Rudolf Carnap, *The Logical Structure of the World. Pseudoproblems in Philosophy* (London: Routledge and Kegan Paul, 1967), pp. 53–54.

[6] In other words, the assumption that every closed sentence takes exactly one of the truth values *true* or *false*. Cf. n. 34 on p. 79 below regarding Pinkard's 'Reply to Duquette'.

Dogmatism of pre-Kantian metaphysics and claims to be dedicated to breaking up its stases.

> The struggle of reason consists in that, to overcome that which the understanding has fixated (*SW* v.8, § 32Z; *Enc*).[7]

This does not necessarily entail a denial of the validity of such reasoning, at least within a limited sphere. Hegel makes a distinction, within thought, between speculative reason and the understanding and he both assigns either-or reasoning to the latter and subordinates it to reason.

> *The theorem of the excluded middle* is the theorem of the determinate understanding which wants to keep the contradiction away from itself, and in so doing commits the very same (*SW* v.8, § 119Z; *Enc*).

The understanding is an essential moment of thought but, and not the least because of formal constraints of its valid exercise, Hegel does not regard it as adequate to philosophical cognition of truth. The mystery for the understanding is its own role.

The distinction between reason and the understanding is taken over from Kant, who had refined and extended use of the term intellect (*Verstand*) in the Wolffian school to apply to the general faculty of cognition as distinguished from *ratio* (*Vernunft*), or the power of seeing the connection between things. Reason, for Kant, as a faculty of *principles*, itself creates concepts, or more strictly Ideas, that are transcendent, that is are not taken from the senses via intuition (*Anschauung*) or from the understanding which gives conceptual unity to intuition through the application of its pure forms, the categories.[8] Hegel's notion of reason, in its departure from Kant, is fundamental to the issue between them. He does not depart from Kant's idea that reason, as the faculty of the unconditioned, that is, thought in its metaphysical exercise, seeks the totality, the unconditioned, the Idea.

> It was only by Kant that the distinction between Understanding and Reason has been pointed out decisively and determined in *such way* that the former has as its object the finite and conditioned, and the latter the infinite and unconditioned. Although it

[7] *Hegel's Logic: Part One of the Encyclopedia of the Philosophical Sciences*, trans. by William Wallace (Oxford: Clarendon Press, 1975). Cited hereafter as '*Enc*'.

[8] Wallace, 'Notes and Illustrations', *Enc.*, p. 310.

> is to be recognized as a very important result of the Kantian philosophy that it has asserted the finitude of the merely empirically based knowledge of the understanding and described its content as *appearance*, it is still not to be stopped at this negative result and the unconditionedness of the reason is not to be reduced to the merely abstract, the difference excluding, identity with itself. ... The same then also applies to the *Idea*, on which Kant did bring back honour insofar as he vindicated it in contrast to the abstract determinations of understanding or even merely sensuous representations (the like of which one may well be in the habit of calling ideas in ordinary life) of the reason, but with regard to which he likewise stopped with the negative and the mere *ought* (*SW* v.8, § 45Z; *Enc*).

Where Hegel *does* depart from Kant concerns the principles for reason's exercise. These principles are the concern of Kant's transcendental logic and of Hegel's dialectical or speculative logic.

2. Hegel Interpretation and Logical Illiteracy

It is one thing for an Hegelian or neo-Hegelian scholar faced, as a philosopher or social theorist without actual competence in modern logic,[9] with the still powerful and still dominant paradigm of classical logic, to take the path of prudence and read Hegel's logic as a form of hermeneutics or as a logic in the broader sense of a method and manner of reasoning.[10] Here at least Hegel's distinction between understanding and reason can be preserved. It is another thing to say that modern logic has ruled out the very value and sense of this distinction as Hegel conceived it, and with that any 'sane' acknowledgement of the critique of understanding or reflective reason from which Hegel's idea of speculative reason proceeds. This second alternative is proposed by Allen Wood. Hegel's ethical theory,

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[9] By which we mean the ability to carry out proofs in logic, not just to cite results.

[10] This position is taken, for instance, in Terry Pinkard, 'A Reply to David Duquette', in *Essays on Hegel's Logic*, ed. by George di Giovanni (New York: New York University Press, 1990), pp. 19–20 (cited hereafter as 'Reply to Duquette'): "First, I want to argue that Hegelian dialectic does not challenge ordinary logic. Second, I want to suggest at least that Hegel's Logic should not to be taken strictly as a logic at all but only as an understanding of philosophical explanation."

in his view, has great merit if only it is taken "as philosophical sanity" judges most promising: in "the understanding's way".

> Viewed from a late twentieth-century perspective, it is evident that Hegel totally failed in his attempt to canonize speculative logic as the only proper form of philosophical thinking. ... When the theory of logic actually was revolutionized in the late nineteenth and early twentieth centuries, the new theory was built upon precisely those features of traditional logic that Hegel thought most dispensable. In light of it, philosophical sanity now usually judges that the most promising way to deal with the paradoxes that plague philosophy is the understanding's way. Hegel's system of dialectical logic has never won any acceptance outside an isolated and dwindling tradition of incorrigible enthusiasts.[11]

It is certainly hermeneutically odd for an interpretation of Hegel's ethical thought to be made within a way of thinking that excludes the understanding of philosophical thought which he was attempting to communicate. The 'understanding's way' is an ambiguous phrase. We would not however, and for reasons which will shortly become apparent, press any norm of hermeneutic method here to the point of proscribing such endeavours. It is not just a curiosity that analytic philosophers keep on producing 'sympathetic' interpretations of writers and texts whose most basic ideas they abhor. If Hegel's idea of the foundations of his philosophical system are dismissed, then so also is his idea of reason and the critique of the understanding on which it rests. This is just to Wood's point. It is part of a politics of the colonisation of metaphysical sense by common sense: a politics that *authorises* itself by allusion to modern logic.

Modern logic, quite simply, does not give this authority. It cannot decide the *philosophical* question that is in issue, namely how logic, metaphysics and ontology are related. Insofar as metaphysics is an inquiry into truth and human capacity for knowledge it includes epistemology. Whether or not it should confine itself to epistemology, leaving ontology more or less aside and deferring questions of ideals or values to a separated exercise of practical reason concerned with moral, legal and political philosophy, is one particularisation of that question. How it is answered

[11] Allen W. Wood, *Hegel's Ethical Thought* (Cambridge: Cambridge University Press, 1990), p. 4. Cited hereafter as 'Wood'.

turns on a set of questions which Wood forecloses. *Do* the logical paradoxes 'plague' philosophy, that is, do they threaten the healthy exercise of reason, or do they threaten the self-satisfaction of the understanding in its blindness to its own role? Might they not be *constitutive* of philosophical thought? One does not have to be Hegelian to acknowledge the latter possibility. It is part of the history of western philosophical thought, a point that has not gone unnoticed by logicians.[12] And since we now draw logicians into the philosophical question, it might be reasonable if not philosophically 'sane', to re-open the question of what 'the most promising way' to deal with the logical paradoxes *is*, from a logically competent perspective.

From such a perspective, Wood's statement needs a triple supplement. In the first place, it is pretentious to talk of precision when it comes to the "features of traditional logic" and that quite independent of Hegel's dealing with them. Even in modern philosophy of logic the issue of what a principle of logic is, is not always sufficiently clear. In particular discussions about non-classical logics are prone to suffer from a lack of awareness in this respect.[13] Apart from that, modern logic extends and revises the

[12] Thus Fraenkel *et al*, set theoreticians, comment on Russell's antinomy. "To be sure, Russell's antinomy was not the first one to appear in a basic philosophical discipline. From Zenon of Elea up to Kant and the dialectic philosophy of the 19th century, epistemological contradictions awakened quite a few thinkers from their dogmatic slumber and induced them to refine their theories in order to meet these threats. But never before had an antinomy arisen at such an elementary level, involving so strongly the most fundamental notions of the two most 'exact' sciences, logic and mathematics." A. A. Fraenkel, Y. Bar-Hillel and A. Levy, *Foundations of Set Theory* (Amsterdam: North-Holland Publishing Company, 1973), p. 2.

[13] 'The principle of bivalence', for instance, is being given up in many ways, usually however without ever asking whether there might be something else that takes its place. The issue here is similar to that of the postulate of the parallels: it may come in a guise that is not readily recognisable as, for instance, a claim about the sum of the angles in a triangle. In some systems of logic, the 'principle of bivalence' can take the guise of any of the following formulas: $A \vee \neg A, \neg(A \wedge \neg A), (\neg A \rightarrow A) \rightarrow A$, $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B), \neg\neg A \rightarrow A$. In fact, replacing I. 3) in the list of formulas in D. Hilbert and P. Bernays, *Grundlagen der Mathematik* I (Berlin, Heidelberg, New York: Springer-Verlag, 1968, zweite Auflage), p. 65, by any of the first three of the foregoing wffs provides an axiomatisation of classical propositional calculus. The fifth and last of the foregoing wffs is characteristic of intuitionistic logic and shows little similarity to what is commonly called *tertium non datur*.
P.S. Since this was first written, we learned from one good referee of the present paper, who was said to have some logical expertise, that intuitionistic logic "does not have

common logic of Hegel's day so extensively that there is no precise mapping between the two.[14] The philosophical question of whether antinomies are to be dealt with in 'the understanding's way', is historically an issue between Kant and Hegel, both of whom were working with the old common logic. That this had hardly altered since Aristotle is a point which both mention. Kant takes it as proof of its soundness. Hegel considers it ripe for revision. Analogies, certainly can be made, but analogies are not precise.

Second, there is a considerable difference between 'dispensing' with features of traditional logic and *restricting* them to thought at the level of the understanding. Taken together these two points persuade some Hegel scholars that it is a myth to say that Hegel denied the 'law of non-contradiction'.[15] We are less concerned with debates concerning Hegel interpretation than with specifying two questions. First, what is involved in, and what kinds of logic come from restricting features of classical logic? Second what is required to read Hegel's *Logic as a logic*? We deal with both questions below, adding a little context from the history of logic, but a preliminary response to the second question, may be helpful here. To read Hegel's *Logic* as a logic, does *not* require commitment to the view that Hegel presented a logical derivation of the categories which has been or could be translated into the formal language of a modern logic. It is rather to see how the occurrence and significance of contradictions in thought that has itself and its own determinations as its objects, lies at the core of Hegel's extension and radicalisation of Kant's transcendental logic.

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excluded middle nor the other formulae listed" here. We have to admit that, in writing this note we did not sufficiently anticipate the possibility of such a response. In face of it our breath is clearly wasted.

[14] *Cum grano salis,* traditional (Aristotelian) logic may be regarded as monadic predicate logic. Cf. J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* (Oxford: Clarendon Press, 1957); also D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, fifth edition (Berlin, Heidelberg, New York: Springer-Verlag, 1967), pp. 57–63); and A. Tarski, *Einführung in die mathematische Logik*, (Berlin: Julius Springer, 1938), p. 46.

[15] Robert Hanna, 'From an Ontological Point of View: Hegel's Critique of the Common Logic' in *The Hegel Myths and Legends*, ed. by Jon Stewart (Evanston, Illinois: Northwestern University Press, 1996), pp. 253–281; Robert Pippin, 'Hegel's Metaphysics and the Problem of Contradiction', *op. cit.*, pp. 239–252.

The third point is more substantial and goes to the analogies between the traditional form of logic and modern mathematical logic that can justifiably be made. It is certainly the case that the 'revolution' in logic that took place in the late nineteenth and early twentieth century, owed nothing at all to Hegel. When the theory of logic was revolutionised, it was certainly not in the spirit of Hegel. It did not question classical logic and, in the work of Gottlob Frege, which made explicit the crucial shift from a concept to its extension, it ran straight into antinomies. These antinomies have come to be known as the *logical paradoxes*. Essentially, they are the kind of paradoxes, such as the Liar, that have been classified as shallow sophistries since the times of Aristotle. Moreover, 'solutions' to the modern paradoxes supplied by modern logic are often considered as artificial and unilluminating, at least by those who favour a different one. This does not vindicate Hegel, but it calls for more caution towards the kind of late twentieth century viewpoint evoked by Wood.

The question, to which Wood never advances, of what the significance of the antinomies into which Frege ran is for logic *and* philosophy, arose *within* logic, within a few decades of the 'revolution' to which Wood refers.

> Logical coercion is most strongly manifested *in a priori sciences*. Here the contest was to the strongest. In 1910 I published a book on the principle of contradiction in Aristotle's work, in which I strove to demonstrate that that principle is not so self evident as it is believed to be. Even then I strove to construct non-Aristotelian logic, but in vain.
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> Now I believe to have succeeded in this. My path was indicated to me by the *antinomies* which prove that there is a gap in Aristotle's logic. Filling that gap led me to a transformation of the traditional principles of logic.
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> ... I have proved that in addition to true and false propositions there are *possible* propositions, to which objective possibility corresponds as a third in addition to being and non-being.[16]

Such was Łukasiewicz' position in 1920. As regards logic, the question was still alive and unsettled more than forty years later:

[16] Łukasiewicz, *Selected Works*, ed. by L. Borkowski (Amsterdam: North-Holland Publishing Company; and Warszawa: PWN-Polish Scientific Publishers, 1970), p. 86.

> the set theoretical paradoxes ... are a very serious problem, not for mathematics, however, but rather for logic and epistemology.[17]

And we find Myhill reaffirming another twenty years later,

> Gödel said to me more than once, "There never were any set-theoretic paradoxes, but the *property-theoretic* paradoxes are still unresolved",[18]

adding that

> the Fregean concept of property is inconsistent *with classical logic*. So if we want to take Frege's principle seriously, we must begin to look at some kind of nonclassical logic.[19]

We leave the assessment of the set theoretical situation to set theorists. Our point is that these comments, from within logic, hardly authorise a cavalier dismissal of antinomies. On the contrary, philosophy might, on pain of signing its own certificate of irrelevance, need further and better details regarding them.

Frege's project was an attempt to reduce arithmetic to higher order logic, that is, in modern terms, the ideal calculus, or classical logic with abstraction axioms. Such axioms, roughly, allow the formation of a concept, (e.g. redness) from a predicate (e.g. is red). While Frege expressed some doubt about whether they satisfied the requirements of purely logical axioms, he confessed himself unable to conceive numbers as objects without them. The problem was (and is) that within classical logic, these axioms lead into antinomies. When this was pointed out to Frege by Russell, Frege regarded his life work as having failed.

> Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished.
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> This was the position I was placed in by a letter of Mr Bertrand Russell, just when the printing of this volume was nearing its completion. It is a matter of my Axiom (V). I have never

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[17] K. Gödel, 'What is Cantor's Continuum Problem?' in *Philosophy of Mathematics*, ed. by P. Benacerraf and H. Putnam (Englewood Cliffs: Prentice-Hall, 1964), pp. 258–273 at p. 262.

[18] J. Myhill, 'Paradoxes', *Synthese* 60 (1984), pp. 129–143 at p. 129.

[19] *Ibid*, p. 130.

> disguised from myself its lack of the self-evidence that belongs to the other axioms and that must properly be demanded by a logical law. And so in fact I indicated this weak point in the Preface to Vol i (p. VII). I should gladly have dispensed with this foundation if I had known of any substitute for it. And even now I do not see how arithmetic can be scientifically established; how numbers can be apprehended as logical objects, and brought under review; unless we are permitted — at least conditionally — to pass from a concept to its extension.[20]

For Frege this was the failure of a project. But what, precisely, did fail, is not clear. In the 'Nachwort' to his *Grundgesetze der Arithmetik* we find Frege pondering over whether the law of excluded middle would have to be restricted, or whether there are cases where we are not entitled to speak of the extension of a concept. Is it asking for too much to see an analogy here between Hegel's and Kant's ways with the Antinomy of Pure Reason?

For modern logic Frege's accomplishment was substantial. It was the accomplishment of what has been called the ideal calculus[21] and this, as noted is *classical* logic, most notably relations and quantifiers, with abstraction axioms.[22] What links Hegel to Frege, or to put that another way, which non-classical logics might be termed dialectical in Hegel's sense of that term, will obviously depend on how Hegel's idea of dialectic is interpreted. In the interpretation presented here, it is abstraction, passing from a concept or predicate to its extension for the purpose of constructing an object of reason: metaphysical reason in Hegel's endeavours, arithmetic in Frege's. As Frege perceived, the failure of his system turned on his *Grundgesetz* V which was introduced to govern the equality of *Werthverläufe*, that is, the extensions of concepts. In the long run, this is what abstraction, as a logical operation, comes down to. Abstraction, in some

[20] P. Geach and M. Black, *Translations from the Philosophical Writings of Gottlob Frege* (Oxford: Basil Blackwell, 1970), p. 234; G. Frege, *Grundgesetze der Arithmetik* v. 2 (Jena: H. Pohle, 1903), p. 253.

[21] For instance, by Fraenkel and others, *Foundations of Set Theory*, above n. 12, p. 154.

[22] More generally, it involved the analysis of the logical components of mathematical reasoning.

cases leads into contradictions and it was just such a contradiction, a 'vicious self-reference' that Russell pointed out to Frege.[23] This, in classical logic and via the rule of excluded middle ($A \vee \neg A$, or some equivalent formulation, cf. n. 13 on p. 68) is a total failure of truth preservation because the contradiction allows anything and everything to be proved.

Somewhat ironically, this situation confronts us with an *either—or* alternative: *either* to preserve classical logic and restrict abstraction (for example, through type distinctions such as Russell proposed in order to find a way around his antinomy) *or* to abandon classical logic and restrict the assumption of truth-definiteness that makes contradiction so unpalatable (in allowing anything and everything to be proved). The first way is probably what Wood means by 'the understanding's way'. It stays within classical logic and restricts abstraction. Of course, "philosophical sanity now usually judges that the most promising way with the paradoxes that plague" Frege's (higher order) logic is to sacrifice the general assumption of the existence of an extension to each and every concept if it has occurred to it that there is a problem. Mathematical logical enterprise is less confined. There are several non-classical logics. All dispense with truth-definiteness, where that is understood as a meta-logical assumption of the validity of either-or reasoning, *as applied to the truth values*, true and false. This is simply what makes them non-classical logics. But they dispense with truth-definiteness in different ways. They might introduce third or further values (as in Łukasiewicz' logics), they might allow cases in which a sentence is both true and false (as in paraconsistent logics, 'dialetheism') or they might aim at allowing unrestricted abstraction by directly dispensing with a 'logical law' in a particular axiomatisation of logic. Such logics may be said to have 'dispensed with' the meta-logical assumption of truth definiteness and might, in light of the link made above, be termed dialectical in an Hegelian sense.[24]

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[23] Strictly speaking, following Frege, it is not possible to predicate a predicate; but via abstraction a predicate can be objectified, and this objectification can then be predicated. For example it makes no sense of any kind to say 'Is red is red' but it can be said 'Redness is red'. In this case, the self-reference brought about by abstraction ('red' predicated of 'redness'), causes no problems. Russell's antinomy concerned the set of all sets which do not contain themselves as elements.

[24] Classical logic *can* be, indeed has been restricted. The possibility of restricting classical logic in such a way as to have unrestricted abstraction available (that is to include Frege's *Grundgesetz* V axiom or its equivalent) is simply not contentious, at

Or they might not. There are all manner of issues and, for that matter, non-issues here. Which non-classical logics *are* dialectical in an Hegelian sense?[25] Are non-classical logics a threat or a complement to classical logic?[26] And then, what does any of this matter? The irony, pointed out above, is that logic cannot take us further with the philosophical questions in issue here: the significance of contradictions in thought that has itself and its own determinations as its objects. The very formality of the either-or of the methods of avoiding Russell's antinomy leaves this question untouched. In that sense we reach a limit of logic's authority. To go further here, in natural language, we must go back to Hegel's issue with Kant — his extension and radicalisation of Kant's transcendental logic — as the classical discussion in modern philosophy on the significance of antinomies in the *a priori sciences*, with what has just been canvassed in mind. That is to say, the problem of antinomies in modern philosophy, while historically an issue between Kant and Hegel, is not *just* an issue between Kant and Hegel and it is not *just* an amusing pastime for speculative philosophers. What Wood does not mention is that 'shallow sophistries', such as the Liar, still plague higher order logic.

This brings us to the point of reading Hegel's *Logic* as a logic. We might recall, to begin with, what Hegel said in his *History of Philosophy* regarding Eubulides' sophisms:

least amongst logicians. It was first established about 1950 independently by Fitch and Ackermann; cf. K. Schütte, *Beweistheorie* (Berlin, Göttingen, Heidelberg: Springer-Verlag, 1960), p. 333 for historical notes, and chapter VIII (pp. 224 ff) for technicalities.

[25] Within a philosophically realist framework, contradictions are located in reality and a non-classical logic that results is dialectical in the sense of dialetheic, that is, it allows that in certain cases, A and ¬A may both be true. See e.g. G. Priest, *Beyond the limits of thought* (Cambridge: Cambridge University Press, 1995), pp. 3 f. This is a philosophy of the limit and is opposed to the more scandalous view that contradictions are not just brute metaphysical facts to which a logic must conform, but are constitutive of the determinations of pure thought.

[26] Some modern logicians, who may be seen as having contributed to the development of a non-classical logic (Kleene and Kripke, for example), remain committed to an idea of truth consistent with classical logic. While working with three 'truth values', the third value ('undefined') is not an *extra* truth value. It is not on the same level as true and false and is not introduced on the assumption that classical logic does not generally hold. Cf. S. C. Kleene, *Introduction to Metamathematics* (Amsterdam: North-Holland Publishing Company, 1952), p. 332 and S. Kripke, 'Outline of a Theory of Truth', *Journal of Philosophy* 72 (1975), p. 700, n. 18.

> The first thing that comes to our mind when we hear them is that they are ordinary sophisms which are not worth refutation, hardly worth listening to them. ... However, it is indeed easier to discard them than to refute them definitively (*SW* v.18, p. 132; *LHP* v.1, p. 457).

Before Gödel, the average philosopher might well have nodded and passed on, still quite content to see only a 'shallow sophistry' in the paradoxes like that of the Liar. Once it is remarked that it was a *variation* of Eubulides' Liar which Gödel employed in his famous incompleteness theorem(s), then it is not unjust to observe that whether someone can only detect a shallow sophistry or a deep epistemological puzzle may well depend on depth of insight.

The antinomies that first prompted Kant to relate logic to metaphysics are not the modern logical paradoxes, although it might be noted that some early set theoretists have pointed out a similarity.[27] But insofar as modern logic can, via careful analogy, throw light on the philosophical question of the relation between logic, metaphysics and ontology, our point is that its discovery of the logical paradoxes, grounds the question so as to open, not close it.

Wood's appeal to Wittgenstein on the most promising way to deal with the logical paradoxes gives an idea of what is likely to be left of Hegel's insight regarding the epistemological significance of the antinomies when that is dealt with in the understanding's way:

> We might compare Hegel's treatment of philosophical paradoxes with the later Wittgenstein's. Wittgenstein held that contradictions or paradoxes do not "make our language less usable" because, once we "know our way about" and become clear about exactly where and why they arise, we can "seal them off"; we need not view a contradiction as "the local symptom of a sickness of the whole body." For Wittgenstein contradictions can be tolerated because they are marginal and we can keep them sequestered from the rest of our thinking; for Hegel, they arise systematically in the course of philosophical thought, but they do no harm so

[27] See e.g. W. Hessenberg, 'Grundbegriffe der Mengenlehre', *Abhandlungen der Fries'schen Schule*, Neue Folge 1,4 (1906), pp. 633 and 706; E. Zermelo in *Georg Cantor, Abhandlungen mathematischen und philosophischen Inhalts*, ed. by E. Zermelo (Hildesheim: Georg Olms, 1966), p. 377 (cited hereafter as 'Cantor').

> long as a system of speculative logic can keep them in their proper place ...[28]

Speculative logic as a special task force, keeping what is considered marginal sequestered from the rest, in its proper place? This wisdom of segregation in the guise of toleration at least suggests that how Hegel's *Logic* is read is not a scholastic issue. In this lies the importance of reading Hegel's *Logic as a logic*. It is logic that is haunted by antinomies. Letting logic off the hook in order to console common sense with Hegel's dialectic may work as an avoidance strategy for Hegelians. Claims such as

> none of Hegel's dialectic in the Logic is in opposition to 'ordinary logic'[29]

and

> Hegelian dialectic is no mysterious form of logic that transcends or is an alternative to ordinary logic.[30]

can survive because there is no sufficiently worked out theory in Hegel's logic, such as, for instance, a theory of arithmetic in the foundational studies of mathematics, that would defy categorical claims of this kind. But nor are such claims warranted. There are only some highly intriguing ideas in a highly difficult (abstract) realm of knowledge, formulated in no less difficult a language. In this situation, by *reversing the focus*, the discovery of the logical paradoxes can serve to open the question as to the nature of Hegel's dialectic. Hegel was not the one who ran unexpectedly into antinomies, it was Frege. In this sense, what is at stake now is higher order logic — not Hegel's idea of dialectic. Frege's logic has failed. The challenge is whether Hegel's idea of dialectic can make a point in the analysis of this failure. Higher order logic has the paradoxes and Hegel's idea of dialectic aims at making sense of contradictions in the enterprise of reason. Higher order logic with its paradoxes, undecidabilities, and incompleteness results is the touchstone of Hegel's idea of dialectic. *Hic Rhodus, hic saltus.*

In so far as Hegel's dialectic endorses a principle of freedom of concept formation (against Kant) it does challenge classical logic; unrestricted

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[28] Wood, p. 3.

[29] Pinkard, 'Reply to Duquette', p. 22.

[30] Terry Pinkard, *Hegel's Dialectic: The Explanation of Possibility* (Philadelphia: Temple University Press, 1988), p. 5.

abstraction is incompatible with classical logic. Those who want to keep Hegel's dialectic in harmony with 'ordinary logic' will have to forgo an unlimited freedom of concept formation.[31] This is not to say that any of Hegel's actual concepts is indeed antinomical. What can be said safely is that Frege's *Grundgesetz* V (or unrestricted abstraction and extensionality) is in opposition to classical logic, in fact already unrestricted abstraction itself is in such opposition. All that is needed to make the link to Hegel is the realisation that unrestricted abstraction is in the spirit of Hegel's speculative philosophy.

To turn this point around: any claim that Hegel's dialectic is not in conflict with classical logic, can only succeed if Hegel can be shown to have proposed a restriction of concept formation to cope with Kantian antinomies. What Hegel does say with regard to Kant's Antinomy of Pure Reason is:

> The main point that has to be remarked is that the Antinomy is not just located in the four particular objects taken from Cosmology, but rather in *all* objects of all kinds, in all representations (*Vorstellungen*), notions, and Ideas. To know this and to recognize objects in this capacity (*Eigenschaft*) belongs to the essential of philosophical consideration; this capacity (*Eigenschaft*) accounts for what furthermore determines itself as the *dialectical* moment of the logical (*SW* v.8, § 48; *Enc*).

What is indeed lacking in Hegel is the actual production of an antinomy, such as the Liar, that would stand up to the standards of modern logic or, at least could be transformed into one. Accordingly the average Hegel scholar can say that, while Hegel may be seen as endorsing a principle which leads to antinomies, this does not mean that these antinomies are 'what he had in mind'. We do not and would not claim any such thing. In fact, we do not see the relevance of such a claim for the problem of a dialectical logic. There is more to that problem than scholastic rereading

[31] We take this to apply to Dieter Henrich's "substantivierte Aussageform" (propositional form turned noun, cf. the first section in his paper "Formen der Negation" in *Seminar: Dialektik in der Philosophie Hegels*, ed. by Rolf-Peter Horstmann (Frankfurt am Main: Suhrkamp, 1978), pp. 213–229) as well, although the endemic lack of precision in philosophical terminology does not allow the establishment of a conclusive link to unrestricted abstraction in logical terms.

of Hegel and while we do not dismiss ongoing efforts of Hegel interpretation, bringing Hegel into relation with modern logic requires competence in modern logic, a point that we find sorely neglected.[32] Unrestricted concept formation produces strange phenomena much stranger than 'table turning' ever was. And the second of the authors wants to add, that these phenomena are not even what he himself dreamt off, when he embarked on the project of making sense of Hegel's idea of dialectic in the framework of higher order logic some thirty-five years ago.

We do not want to close this section on "Hegel interpretation and logical illiteracy" without having produced at least one example of what we consider a fine alternative to a poor 'late twentieth-century perspective':

> Two of the greatest logico-mathematical discoveries of fairly recent times may in fact be cited as excellent and beautiful examples of Hegelian dialectic: I refer to Cantor's generation of transfinite numbers, and to Goedel's theorem concerning undecidable sentences. In the case of Cantor we first work out the logic of the indefinitely extending series of inductive, natural numbers, none of which transcends finitude or is the last in the series. We now pass to contemplate this series from without, as it were, and raise the new question as to how many of these finite, natural numbers we have. To answer this we must form the concept of the first transfinite number, the number which is the number *of* all these finite numbers, but is nowhere found *in* them or among then, which exists, to use Hegelian language, *an sich* in the inductive finite numbers, but becomes *für sich* only for higher-order comment. And Cantor's generation of the other transfinite numbers, into whose validity I shall not here enter, are all of exactly the same dialectical type. Goedel's theorem is also through and through dialectical, though not normally recognized as being so. It establishes in a mathematicized mirror of a certain

[32] It was not an analysis of Leukippos' and Demokritos' writings which substantiated any claim about atoms; it was not a *theory* of atoms that was handed down to us from the ancient Greeks, but an intriguing *idea*. Like every good idea there comes a time when one can do something with it. Hegel's idea of dialectic is just such a good idea to remember when confronted with the situation of higher order logic.

syntax-language that a sentence declaring itself, through a devious mathematicized circuit, to be unproveable in a certain language system, is itself unproveable in that system, thereby setting strange bounds to the power of logical analysis and transformation. But the unproveable sentence at the same time soars out of this logico-mathematical tangle since the proof of its unproveability in *one* language is itself a proof of the same sentence in *another* language of higher level, a situation than which it is not possible to imagine anything more Hegelian.[33]

3. Basic Ideas of Dialectical Logic

What we have said so far would remain as futile as any of those logically illiterate claims and polemics for or against a dialectical logic challenging classical logic, if we were not to give some indication as to what we propose as a dialectical logic, that is, a logic that does not require us "to frame determinations of things in terms of either/or propositions".[34] But we do not want to be misunderstood: the issue is too complex to be dealt with

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[33] J. N. Findlay, 'The Contemporary Relevance of Hegel', in *Hegel. A Collection of Critical Essays*, ed. by A. MacIntyre (Notre Dame and London: University of Notre Dame Press, 1976), p. 6 f.

[34] By speaking of 'determinations of things in terms of either/or propositions' we mean determinations of things in terms of 'either x or not x', not any arbitrary x and y, i.e. 'either x or y', like, for instance: my computer is either made in Australia or standing on my desk. This remark is necessary in view of Pinkard's 'Reply to Duquette', (p. 20): "[Mr Duquette] says that ordinary logic requires us to frame determinations of things in terms of either/or propositions ... But logic *per se* does not require me to put things into either/or dichotomies; just note that the truth table for 'x or y' is different from the truth table for 'either x or y'." The truth table for 'either x or not x' is the same as that for 'x or not x'. Having said this, we hasten to emphasise that trivia of that kind are not the issue of the present section. What is the issue of the present section is that the identification of a logical constant with its truth table misses the point of an alternative logic altogether. In more technical terms, the message of the present section is that logic manifests itself in the so-called structural rules of a Gentzen-type formulation of logic. These structural rules regulate our dealing with assumptions, and this makes a difference to how the truth table of 'or', for instance, acts logically.

conclusively within the limited space of a paper of this kind; all we try to do is to evoke some interest and give some hints.[35]

Before turning to the more technical aspects, we want to try, at least, to convey some basic understanding of the issue in question. For that purpose, consider the following statements taken from different authors:

Hegel: To the ordinary (i.e. the sensuous-understanding) consciousness, the objects of which it knows count in their isolation for independent and resting on themselves.[36]

Cantor: [What we deal with in set theory are] manifolds of unconnected objects, i.e. manifolds of such a kind that removing any one or more of their elements has no influence on the remaining of the others.[37]

Wittgenstein: Each item can be the case or not the case while everything remains the same.[38]

Harris: The fundamental algebraical laws ... of commutation, association, and distribution ... hold only ... for entities that are externally related or are composed of externally related elements.[39]

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[35] Readers who want to know more regarding the mathematical logical side of what we propose as a dialectical logic are referred to: U. Petersen, 'Logic Without Contraction as Based on Inclusion and Unrestricted Abstraction', *Studia Logica* 64 (2000), pp. 365–403.

[36] *SW* v.8, § 45Z; *Enc.*

[37] Zermelo (ed.), 'Cantor', p. 470, n. 2; (our translation).

[38] Ludwig Wittgenstein, *Tractatus logico-philosophicus* (London: Routledge & Kegan Paul, 1969), p. 7.

[39] E. E. Harris, *Formal, Transcendental, and Dialectical Thinking* (Albany: State University of New York Press, 1987), pp. 32–33. This quotation is brutally edited to make it fit in with the other ones, although, we believe, it is not distorting. It is worthwhile, however, to quote a little more within the edited passage since it conveys, to our minds, an understanding of dialectical thinking that comes extremely close to our own. "If ... the units that made up a collection were internally related so that they affected one another in certain ways or constituted one another by their mutual relations, if, in short, we were dealing with wholes and not with mere collections, the order in which the elements were aggregated would not be indifferent and the algebraic laws would no longer hold." (*Ibid.* p. 33.) The emphasis, for us, lies on "the order ... would not be indifferent", and this is what we aim at by focusing on the structural rules below: roughly, the structural rules do for propositions (in logic) what the algebraic laws do for externally related objects, such as numbers (in arithmetic).

What shines through in these quotations, despite the differences in their claims, is an awareness of a possible alternative: are the objects that we are dealing with isolated things that have their properties independent of what anything else does around them, including our knowledge of them; or is ours a world of interconnectedness where it is in principle never possible to isolate an object, not even in thought?

This raises two questions. Firstly, why are the objects that we want to take into account in dialectical logic not severally independent? Differently put: what is there to relate entities internally, as distinct from externally? Secondly, how does classical logic have to be adjusted (if at all) in order to deal appropriately with objects which are internally related, or inherently connected?

Our answer to the first question in a nutshell: because conceptual thought is constitutive for all knowledge, and conceptual thought has the inescapable double character of form and content which manifests itself in an original ambiguity.[40]

This answer is derived from an analysis of Gödel's first incompleteness theorem, an analysis which cannot be presented here in full, though we shall try to give the gist of it.

Gödel's (formally) undecidable sentence involves a certain substitution function *sub* which satisfies the following condition

$$sub(\ulcorner \mathfrak{A}[x] \urcorner, n) = \ulcorner \mathfrak{A}[n] \urcorner,$$

where the little corners $\ulcorner\ \urcorner$ indicate the well-known device of numerical codification that Gödel introduced in his famous paper of 1931;[41] $\mathfrak{A}$ is a so-called nominal form,[42] a metatheoretical device for communicating that any well-formed expression of the language in question with certain indicated 'empty places' in which the expression in square brackets following it is to be inserted, may take its place; more intuitively, perhaps, any propositional form can be substituted for it. In plain words the above

[40] Dubbed *systemic ambiguity* by the second author. Cf. footnote 45 below.

[41] 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatshefte für Mathematik und Physik*, 38, pp. 173–198. Translated as 'On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems' in J. van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic* (Cambridge, Massachusetts: Harvard University Press, 1967), pp. 596–616.

[42] Cf. Schütte, *Proof Theory* (Berlin, Heidelberg, New York: Springer-Verlag), p. 11.

equation reads: $sub(\ulcorner\mathfrak{A}[x]\urcorner, n)$ equals the Gödel number of the result of replacing every indicated occurrence of x in $\mathfrak{A}[x]$ by the numeral n. Gödel's trick consists in taking for both arguments of this substitution function the Gödel number of the expression $\mathfrak{A}[sub(x,x)]$, i.e. $\ulcorner\mathfrak{A}[sub(x,x)]\urcorner$. Let us take $\mathbf{k}_{\mathfrak{A}}$ as an abbreviation for $\ulcorner\mathfrak{A}[sub(x,x)]\urcorner$ and we obtain the following (indirect) 'fixed point property':

$$sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}}) = \ulcorner\mathfrak{A}[sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}})]\urcorner .$$

The reason that this is called a "fixed point property" should become sufficiently clear when we take the abbreviation $f_{\mathfrak{A}}$ for $sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}})$:

$$f_{\mathfrak{A}} = \ulcorner\mathfrak{A}[f_{\mathfrak{A}}]\urcorner$$

and call $f_{\mathfrak{A}}$ a fixed point with regard to $\mathfrak{A}$: if $\mathfrak{A}$ is regarded as a propositional function, then its value for the argument $f_{\mathfrak{A}}$ is $f_{\mathfrak{A}}$ itself. Such a fixed point property causes trouble for the expressibility of basic semantical concepts on the level of the formalised theory itself (i.e. as an arithmetical predicate, such as, for instance, the predicate of being a prime number), most notably that of truth, i.e. a predicate that satisfies the following 'truth condition':

$$tru(\ulcorner A\urcorner) \leftrightarrow A .$$

To see this, assume the existence of such a predicate *tru*. Obviously it satisfies

$$tru(f) \leftrightarrow tru(f) ,$$

and by the above fixed point property there is a fixed point $f_{\neg tru}$ such that:

$$f_{\neg tru} = \ulcorner\neg tru(f_{\neg tru})\urcorner .$$

By the substitutivity of equal numbers in arithmetic propositions these two yield:

$$tru(f_{\neg tru}) \leftrightarrow tru(\ulcorner\neg tru(f_{\neg tru})\urcorner) .$$

On the other hand, by the above truth condition, one has

$$tru(\ulcorner\neg tru(f_{\neg tru})\urcorner) \leftrightarrow \neg tru(f_{\neg tru}) .$$

By the transitivity of $\leftrightarrow$, the last two yield:

$$tru(f_{\neg tru}) \leftrightarrow \neg tru(f_{\neg tru}) ,$$

i.e. an antinomy.[43]

What happens — in the establishment of the (indirect) fixed point property which lies at the bottom of these results — is that we have (the formal representative of) a number here, which we called $\mathbf{k}_{\mathfrak{A}}$, which occurs as the argument of the function *sub* in two different roles. One time it occurs as an innocent number, i.e. it is being constructed from 0 in a series of steps of adding 1. The other time, however, it occurs as a hieroglyphic behind which a complex proposition is hiding. The substitution function juggles with these two sides of $\mathbf{k}_{\mathfrak{A}}$, which accounts for the curious double character in the employment of $sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}})$, and according to which way we look at this number, we get conflicting results. This is what we take as our paradigm of a conflict between form and content.

In other words, Gödel's construction of a formally undecidable sentence involves a mathematically immaculate form of a use-mention confusion.[44] This confusion is the source of a certain ambiguity which is inescapable once a sufficient amount of arithmetic is available. It provides the answer to our first question. Differently put: the understanding's way, governed by the silent assumption that the objects of our thought can be treated as severally independent, unconnected, externally related, is incompatible with the actual existence of a connection, an internal relation, provided by Gödel's encoding.[45]

This confusion does no harm, as long as there are no semantical concepts available which would be sufficient to establish a connection between the formal system and its intended interpretation, like that of truth or

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[43] Readers who 'find themselves puzzled' in some of the logical moves involved in this reasoning may find it comforting to know that the technicalities do indeed require some basic skill in mathematical logic, in the absence of which the correctness of these moves would have to be taken on trust. We refer to our footnote 9 above. Readers with more serious ambitions might find it helpful to consult a survey article such as C. Smoryński, 'The Incompleteness Theorems', *Handbook of Mathematical Logic*, ed. by J. Barwise (Amsterdam: North-Holland Publishing Company, 1977), in particular, pp. 826–7. A condensed treatment can also be found in G. Takeuti, *Proof Theory* (Amsterdam: North-Holland Publishing Company, 1987), in particular, pp. 82–85.

[44] R. L. Goodstein, *Essays in the Philosophy of Mathematics* (Leicester: Leicester University Press, 1967), p. 20: "The code has been used and mentioned, and there is no self-reference."

[45] In U. Petersen, *Diagonal Method and Diagonal Logic* (Osnabrück: Der Andere Verlag, 2002), section 111d, p. 1530, the label "systemic ambiguity" is introduced for this phenomenon.

satisfaction, for instance. It is only the source of incompleteness and undecidability results. One small step, however, and hell breaks loose: add a sentence which is provable in a meta-theory, like that of the consistency of the object-theory in question, and everything becomes provable. The classic example is that of a reflection principle for the provability predicate of first order arithmetic, provable in second order arithmetic,[46] but incompatible within first order arithmetic itself. Such is the situation of theories based on classical logic, in which a certain amount of arithmetic is available.

We thus come to our second question: how can we take account of the internal relatedness of our objects? Differently put: how can we avoid the implicit assumption of the understanding's way that objects are severally independent? How does an assumption of several independency manifest itself on the logical level? Is logical reasoning possible without the assumption that the objects of our thought are severally independent?

This is a tricky question, or rather cluster of questions, because it more or less implicitly requires an answer to the question: what is logic? Or, at least, what is the difference between classical and non-classical logics?

Our answer to this question is derived from some well-established techniques within proof theory, a familiarity which, unfortunately is hardly to be found amongst philosophers in the Hegelian tradition, and only little more amongst philosophers in the analytic tradition. These techniques are linked to the name of Gerhard Gentzen and their central features are cut elimination and normalisation.

In 1934, Gentzen proposed a formulation of classical and intuitionistic logic in terms of so-called *sequents* ("Sequenzen").[47] We shall restrict our attention here to the case of intuitionistic logic, since it is slightly simpler to present while it shows, at the same time, all the relevant features required to make our point.

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[46] This simply says: if $\ulcorner A \urcorner$ is the Gödel number of a provable formula A, then A; less technical: if A is provable, then A.

[47] Gerhard Gentzen, 'Untersuchungen über das logische Schließen', *Mathematische Zeitschrift*, 41 (1934), pp. 176–210 and 405–431. Translated by M.E. Szabo in *The Collected Papers of Gerhard Gentzen* (Amsterdam and London: North-Holland Publishing Company, 1969).

A sequent has the following form

$$A_1, \ldots, A_n \Rightarrow C\ ,$$

where $A_1, \ldots, A_n, C$ are formulas. The formulas left of $\Rightarrow$ are considered assumptions, the formula on the right of $\Rightarrow$ the hypothesis. Rules in Gentzen's formulation of logic are divided into two kinds: structural rules and operational rules. The rules for handling logical constants are the operational rules. In the case of "or", in symbols $\vee$, they look like this (where Γ and Π denote sequences, as distinct from sequents, of formulas, such as $A_1, \ldots, A_n$, for instance):
Introduction left:

$$\frac{A, \Gamma \Rightarrow C \qquad\qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ .$$

Introduction right:

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \text{and} \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ .$$

These rules perfectly mirror the truth values if one takes a sequent to be true if one of the assumptions is false, or the hypothesis is true. They do not, however, fully determine the meaning (or behaviour) of the disjunction "or". What is needed in addition are rules which regulate the handling of the assumptions:
Weakening

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ .$$

Exchange

$$\frac{\Gamma, A, B, \Rightarrow C}{\Gamma, B, A, \Rightarrow C}\ .$$

Contraction

$$\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ .$$

In words: weakening says that assumptions may be added according to taste, exchange says that the order of two assumptions may be reversed, and contraction says that having an assumption once is as good as having it twice, or as Girard put it:

> *contraction* is the fingernail of infinity in propositional calculus: it says that what you have, you will always keep, no matter how you use it.[48]

Note that these structural rules involve no logical constants. Nevertheless, they are the true backbone of classical logic. As Girard put it:

> these rules are the most important of the whole calculus, for, without having written a single logical symbol, we have practically determined the future behaviour of the logical operations.[49]

And:

> It is not too excessive to say that a logic is essentially a set of structural rules![50]

One example in which the future behaviour of the logical operation $\vee$ ('or') is determined by the structural rules is *tertium non datur*, $A \vee \neg A$. Without contraction it is impossible to obtain *tertium non datur* from the above operational rules for $\vee$.

In the light of these considerations regarding the role of assumptions in logic, we can now formulate our answer to the second question: because of the double character of concepts, two occurrences of the same statement in a proof may not without further provision be assumed to have the same truth-value, i.e. we look at formulas in logic as *tokens* and not types. This view of formulas as tokens can be incorporated in Gentzen's formulation of logic by dropping the rule which allows the reproduction of assumptions *ad libitum*: contraction.[51] This idea was put forward in 1980 (by the second author):

> Having inferred B from A and A → B we cannot expect ... that A and A → B are still available as presuppositions (assumptions). It is possible that they have changed in the process of inferring,

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[48] J.-Y. Girard, 'Towards a Geometry of Interaction', *Contemporary Mathematics*, 92 (1989), pp. 69–108 at p. 78.

[49] J.-Y. Girard, Y. Lafont, P. Taylor, *Proofs and Types* (Cambridge: Cambridge University Press, 1989), p. 30.

[50] J.-Y. Girard, 'Towards a Geometry of Interaction', p. 78.

[51] This is not to be confused with *adding* assumption; that's what weakening does. Contraction allows assumptions to be used more than once and in that sense it allows the reproduction of assumptions; or, if you prefer: multiplication of resources at no extra costs.

> that they have been exhausted, so to speak. This means we interpret the implication A → B as "A transfers into B". In this way we want to take account of the peculiarity of unrestricted abstraction.[52]

In this sense, dialectical logic is a resource conscious logic,[53] a logic in which attention is paid to the manipulation of assumptions. Classical logic has no space for a dynamics of assumptions: the structural rules override it; truth and falsity is determined before we start reasoning. Reasoning under the rule of classical logic can only establish truth *for us*; it is subjective in the sense that the objective state of affairs is determined before we start reasoning. Classical logic cannot allow reasoning to be part of the truth, and in so far as the paradigm of classical logic is the understanding's world, truth can never reside in thought determinations.[54] Classical logic has no truth within itself; it can only be truth preserving, never generating.[55]

4. Dialectical Thought versus Finite Thought — the Example of the Complement

Having fixed a logic which does not succumb to either-or reasoning in the specific sense that unrestricted abstraction is allowed without causing 'head-on contradictions' ("kontradiktorische Widersprüche"), there is still the question of what that actually means for logical reasoning.

It will perhaps be clear that the difference between dialectical thought and classical thought is subtle and just as the structure of the cell does

[52] U. Petersen, *Die logische Grundlegung der Dialektik* (München: Wilhelm Fink Verlag, 1980), p. 97; (our translation).

[53] The term is taken from A. S. Troelstra, *Lectures on Linear Logic* (Stanford: Center for Studies of Language and Information, 1992), p. 1. In the past ten years one particular specimen of a resource conscious logic has had a major impact on computer science, the linear logic of J.-Y. Girard.

[54] "The question regarding the truth of the thought determinations must seem strange to the ordinary consciousness ... This question, however, is just what matters (*worauf es ankömmt*)" (Hegel *SW* v. 8, § 24Z(2); *Enc.*).

[55] This has to be contrasted with the following: "Hegel was also worried about logic's formality, since he thought it doubtful that logic could be 'true' if it were purely formal. He could have avoided that worry altogether if he had been in the position to hold the contemporary view that logic is not intended to *provide* truth at all but just to *preserve* it." Pinkard, 'Reply to Duquette' at p. 23.

not reveal itself to the naked eye, the subtleties of dialectical thought do not reveal themselves to plain thinking. The most spectacular aspect of unrestricted abstraction is a so-called (direct) *fixed point property.*[56] What it says in plain words is that to every concept, the list $\mathfrak{F}$ of properties of which contains occurrences of y, there is an object f, the *fixed point* of $\mathfrak{F}$, such that a replacement of these occurrences of y by occurrences of f results in a concept which equals f. Since this will make the head of a logician go into a spin, we add a formulation in the artificial language of symbolic logic:

$$\lambda x\, \mathfrak{F}[x, f] = f\ .$$

This gives rise to a beautiful example of a theorem in classical logic which no longer prevails in its original form in dialectical logic (as outlined above). It can be found in Leibniz in the following form (including a proof):

> Theor. X.
>
> *Detractum et Residuum sunt incommunicantia.*
>
> Si L – A∞N, dico A et N nihil habere commune. Nam ex definitione detracti et Residui omnia quae sunt in L manent in N praeter ea quae sunt in A, quorum nihil mane in N.[57]

In modern set theory it runs (without a proof)

A set and its complement are disjunct.

In set theoretical symbolism:

$$\mathrm{M} \cap \complement(\mathrm{M}) = \varnothing\ ,$$

where $\complement(\mathrm{M})$ is the complement of M and $\varnothing$ is the empty set. In other words: the intersection between a set and its complement is empty. Or: M et $\complement(\mathrm{M})$ nihil habere commune.

This touches on an extremely delicate and crucial point. Is it possible, in principle, to divide the world into two disjunct parts, the union of which is the world, i.e. is it possible to have a division of the world

[56] This is to be distinguished from the indirect fixed point property from p. 82, insofar as the fixed point is not hidden within the little corners ⌜ ⌝. Labelling fixed points 'direct' and 'indirect' is not common in logic; it suggests itself for logicians who want to accommodate for unrestricted abstraction.

[57] G. W. Leibniz, *Schriften zur Logik und zur philosophischen Grundlegung von Mathematik und Naturwissenschaft*, ed. by Herbert Herring (Frankfurt am Main: Suhrkamp, 1996), p. 170.

without remainder? The classical logician has provided an answer before the philosopher comes on the scene: *tertium non datur* does just that.

If the classical logician is right, there is no room for Hegel's dialectic. All that might be possible is a diluted form like a hermeneutics of categories. But then, if the classical logician is right, there is also no room for unrestricted abstraction, because unrestricted abstraction (with some basic logic) provides the (direct) fixed point property. And what the (direct) fixed point property for terms tells us is that there is an element f (a 'fixed point') such that $\mathsf{C}(f) = f$. This has a decisive impact on the above theorem: on the one hand, we have

$$f \cap \mathsf{C}(f) = \varnothing$$

by the theorem, and on the other hand

$$\mathsf{C}(f) = f$$

by the fixed point property, i.e.

$$f \cap f = \varnothing$$

by substitutivity of equals. In words: the intersection of f with itself is empty. In classical set theory this means that $f = \varnothing$, i.e. f itself is empty; but then, the complement of the empty set is the universal set. From a classical position this leaves no choice but to exclude the fixed point f as unpalatable. This is what logicians have mostly done since Russell's discovery of his antinomy. The decision that weird terms such as Russell's class have to be avoided has been handed down to philosophers of somewhat Hegelian persuasion. But when modern logic finally arrives at the level of philosophers it has been reduced to a heap of dead bones not much different in character to those that Hegel saw in the logic of his time.

So what is wrong in Leibniz' reasoning, or the reasoning of modern set theory, from a dialectical point of view? The answer is that it does not take into account the role of assumptions in the reasoning related to notions of 'incommunicantia' or 'disjunct'; more specifically to the notion of 'and' that is involved in these concepts.

In the absence of contraction the classical truth tables for conjunction do not fully determine just one particular notion of conjunction. As a consequence, dialectical logic distinguishes two forms of intersection: $\cap$ and $\sqcap$; relying on the two different notions of conjunction. Both notions of conjunction are characterised by the same (classical) truth values. What distinguishes them is the handling of assumptions.

Between them the two notions of conjunction divide all the properties that their classical counterpart combines in one. Leibniz' theorem, for instance, does indeed hold for the one form of intersection, communicated by $\sqcap$:

$$\mathrm{M} \sqcap \complement(\mathrm{M}) = \varnothing\,;$$

but what fails is $f \sqcap f = f$. For the other form of intersection, communicated by $\cap$ the situation is exactly the other way round.

This situation gives rise to a variation on an eminently Hegelian theme, the identity and non-identity of being and nothing. What can be established with the help of the fixed point property is that to every concept there exists another one, a 'doppelgänger' as it were, which is equal but not identical to the original one, i.e. any object that falls under one of them also falls under the other. Still, they are not the same in the following sense: in so far as they may be regarded as objects themselves, they have different properties, i.e. they cannot be substituted for each other regardless of context.

In Hegel's (translated) words:

> Their difference is ... completely empty ...; it thus does not subsist in themselves, but only in a third, in *opinion* (*SW* v.4, p. 101; *SL*, p. 92).

Contraction free logic with unrestricted abstraction has space for a phenomenon of this kind; in fact, it creates such phenomena in abundance. They are the mysteries of the understanding, and their presence calls for another sacrifice on the part of the classical doctrine: 'extensionality' is just the principle that if two concepts subsume the same objects under them, then they may be substituted for each other *salva veritate*.[58] This principle of identity, an integral part of Frege's logic in the *Grundgesetze*, is incompatible with the possibility of unrestricted abstraction in higher order logic. This is the more remarkable as Frege's celebrated distinction

[58] Thus Leibniz defined: "*Eadem* sunt quorum unum potest substitui alteri salva veritate." G. W. Leibniz, *Schriften zur Logik und zur philosophischen Grundlegung von Mathematik und Naturwissenschaft*, p. 156. ("Those terms are 'the same' of which one can be substituted for the other without loss of truth." *Leibniz. Logical Papers. A Selection*, ed. and trans. by G. H. R. Parkinson (Oxford: Clarendon Press, 1966), p. 123).

of sense and reference was, and that not in the last instance, meant to provide support for extensionality, at least in logic and arithmetic.[59] To paraphrase Hegel:

> There is mystery in higher order logic, only however for the understanding which is ruled by the principle of abstract identity.

Or, as someone by no means less famous than Hegel has not quite said some time before Hegel:

> There are more things in higher order logic,
> Than are dreamt of in understanding's philosophy.

[59] "I use the word "equal" to mean the same as "coinciding with" or "identical with"; and the sign of equality is actually used in arithmetic in this way. The opposition that may arise against this will very likely rest on an inadequate distinction between sign and thing signified." *Gottlob Frege — The Basic Laws of Arithmetic. Exposition of the System*, trans. by M. Furth (Berkeley, Los Angeles: University of California Press, 1964), p. 6.