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Inhaltsverzeichnis

Vorwort der Herausgeber ix

Defining the Precautionary Principle: Uncertainties and Values in Science for Policy
MATTHIAS KAISER 1

On Re-staging the Universal: Butler, Hegel and Contesting the Closure of Logic
VALERIE KERRUISH 12

Philosophical Sanity, Mysteries of the Understanding, and Dialectical Logic
VALERIE KERRUISH AND UWE PETERSEN 29

Some Additions and Corrections to *Diagonal Method and Dialectical Logic*
UWE PETERSEN 43

Vorwort der Herausgeber

Unter dem Titel „DILEMMATA“ erscheinen die Jahrbücher der „Altonaer Stiftung für philosophische Grundlagenforschung“ (ASFPG), von denen wir hiermit den 1. Jahrgang vorlegen.

Die ASFPG widmet sich satzungsgemäß der interdisziplinären Grundlagenforschung in den Bereichen Logik, Epistemologie, Mathematik, Metaphysik, angewandter Ethik und Recht. „Interdisziplinär“ bedeutet dabei vor allem, daß der Trennung von Natur- und Geisteswissenschaften entgegen gearbeitet werden soll. Die ASFPG versucht bewußt, sich von akademischer Kleinstaaterei abzusetzen.

Die Arbeit der Stiftung ist auf drei Zentren verteilt, ein Entwicklungszentrum für spekulative Logik, ein Zentrum für Rechtstheorie, und ein Zentrum für Umwelt- und Technologie Ethik. Diese Dreiteilung repräsentiert die Arbeitsgebiete der drei Vorstandsmitglieder der ASFPG, die sich hier, in diesem ersten Jahrbuch, mit ihren Beiträgen vorstellen.

Das Jahrbuch der Altonaer Stiftung für philosophische Grundlagenforschung soll einen Einblick in die Arbeit der Stiftung geben. Neben Beiträgen der Vorstandsmitglieder sollen in Zukunft auch Arbeiten, die im Rahmen des Stiftungszwecks und/oder mit Geldern der Stiftung erstellt wurden, im Jahrbuch veröffentlicht werden.

Der Vorstand der ASFPG
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Defining the Precautionary Principle: Uncertainties and Values in Science for Policy*

Matthias Kaiser

The French aristocrat, philosopher, scientist and statesman Condorcet was perhaps one of the earliest and strongest protagonists of the belief in progress writ large. In his last book ***Sketch for a Historical Picture of the Progress of the Human Mind*** *(1795) he lays down his optimistic vision of the progress of man, both past and future. The book was written under great strain while already in hiding from the Revolution that Condorcet once supported and that now had turned against him. In the end he asks the question (that later Malthus is asking as well) whether increased welfare and improved health of man will lead to largely increased populations – and, if population increases, will not necessarily there be a time when the number of people has outgrown the natural resources that nature can supply? And is it not reasonable to assume that when resources become scarce, then there will be fight for the resources, war between people, just the opposite of his vision of progress? Condorcet has two answers in stock to this challenge.*

Firstly, nobody could claim that such a time is imminent (written in 1794), it is assumedly far into the future. And nobody can know what technological progress might have achieved at that time. Technology might have the answer in store. This is Condorcet's technology-fix argument. Secondly, he argues that once humankind has progressed that far by means of knowledge and technology, one must assume that also people's ethics and morality has progressed alongside reason. And then it must be clear that our moral duty is not to make sure that unborn life is born, but that those that are born are secured a life in reasonable welfare, dignity and happiness. For Condorcet, the progress of knowledge and technology is unthinkable without implying a parallel progress of human morality. This is Condorcet's ethics argument.

* Parts of this paper are based on previously published work and taken from Kaiser (2003, 2004, 2005). Other parts are taken from the preparations to the report UNESCO/COMEST 2005, in which the author was involved as chair of the expert group.

Yet, it is precisely this coupling of scientific/technological progress with a matured sense of morality and ethics that is questioned by many people at end of the 20^{th} and the brink of the 21^{st} century, two-hundred years after Condorcet. People ask whether technology is out of control, a run-away train without steering and aim. And people ask whether our science no longer feels a commitment to serve the public good but has become the servant of powerful interests, benefiting only a few and risking the harm of many? Have we developed the right moral attitudes and instruments to manage the risks that science and technology produces? Has innovation lost sight of solidarity and neglected the challenge of socially desirable ends?

For a long time scientific progress was seen as exclusively being indebted to so-called epistemic values, i.e. increasing our knowledge about the world. Science as such was deemed to be essentially value-free. But science and the technology following it has changed our life-world in many ways, more rapidly than ever before in history. This has given rise to new questions and challenges.

The belief in social progress reached its peak some time during the 19^{th}century, arguably most vividly expressed in the 1851 world exhibition in London and the building erected for this purpose: the Crystal Palace. But it did not last. The sinking of the Titanic in 1912 was a foreboding of the limits of technological control. Gas warfare during WWI sent a signal that science not only has the capacity to produce inhuman technology, it also showed that such technology will be used. The atom-bombs that were released over Hiroshima and Nagasaki at the end of WWII were the result of intense scientific research (the "Manhattan project") and they raised the same worry: will science and technology turn out to be more of a threat to humanity than a blessing? When the book by Rachel Carlson *Silent Spring* came out during the 1960's it apparently documented how what was originally perceived as scientific breakthroughs later turned out to be a big environmental problem. This was the Janus-face of scientific progress: every benefit that resulted from science and technology seemed to be coupled to the downside of producing new problems as unintended side-effects. The belief in progress was shattered or at least perceived with ambiguity. In the mind of the public, including some of the political decision makers, something needed to be done to correct these negative consequences of science and technology.

Two things resulted from this:

1. a call for more ethical responsibility in science and technology
2. a new generation of environmental regimes that aimed at controlling or managing the consequences of human interaction with the environment.

In the following I shall make the claim that the celebrated Precautionary Principle (hereafter abbreviated as PP) can be understood as combining these two trends. I shall try to elaborate what the PP is, what it implies and how it is justified.

Caring for the environment by different regimes

The early stages of national and international environmental policies can be characterised by a *curative* model of our natural environment: with increased environmental impacts of growing populations and industrialisation, the environment could no longer cure itself; it should thus be helped to repair the damage inflicted upon it by human activities. For reasons of equity and feasibility governments sought to apportion the economic costs of such intervention by requiring polluters to pay the cost of pollution. It soon became apparent, however, that this *Polluter Pays Principle* was practicable only if accompanied by a preventive policy, intended to limit reparation to what could be compensated. This 'prevention is better than cure' model marks the second stage of governmental action for environmental protection. This stage was characterised by the idea that risks are known and quantifiable, and the *Prevention Principle* guided policy making. This was the heyday of quantitative risk assessment and risk-cost-benefits analyses. The emergence of increasingly unpredictable, uncertain, and unquantifiable but possibly catastrophic risks such as those associated with GMOs, climatic change etc., has confronted societies with the need to develop an additional third, anticipatory regime to protect humans and the environment against unanticipated risks of (new) technologies: the *Precautionary Principle* or 'better safe than sorry' model. The emergence of the PP has marked a paradigmatic shift from *a posteriori* control (civil liability as a curative tool) to the level of *a priori* control (anticipatory measures) of risks (de Sadeleer, 2002).

Over the past decades, the PP has become an underlying rationale of a large and increasing number of international treaties and declarations in the fields of *inter alia* sustainable development, environmental protection, health, trade, and food safety. The PP is on its way to become a widely accepted part of international law. In its basic form, the PP states that action to protect human health and the environment to avoid possible danger of severe and irreversible damage, need not wait for rigorous scientific proof (Weiss, 2003). In practice, different and somewhat diverging formulations, definitions and interpretations of the PP can be found. Further, a multitude of contradicting perspectives of what makes up a precautionary approach coexist amongst major players in the international arena.

The PP forms a meeting ground of tremendous tensions: between supra-national and national legal orders, between the global and the local, between law and science, between North and South, and between certainty based 'positivist' views of science and uncertainty based 'post-modern' and 'post normal' interpretations of science (Funtowicz & Ravetz 1992).

Thus, some see the PP as essentially anti-scientific, anti-rational, anti-innovation, anti-sustainable use, or Northern in outlook. Others defend it as an ethically founded principle for responsible co-existence in a globalised context, as a safeguard to care for future generations, as integral to sustainable development, as truly responsible science. Much of the debate has focused on the use or abuse of the PP in international trade where some fear it may be used as a new instrument for trade barriers, while others stress that the PP provides the assurance to Nation States that their chosen levels of safety will not be compromised by international trade.

In discussing the PP one needs to be aware of four different contexts which must be understood as relevant background for the complex discussions about PP. These contexts are: 1. the scientific context; 2. the legal context; 3. the political context, and 4. the ethical and cultural context. In the following sections we shall not have the space to discus all of these aspects in detail.

1. The *scientific* context: It emerged early that some scientists, while embracing the principal ideas of precaution, assumed it had no repercussions on science, and would leave science basically unaffected. The PP was seen as a principle for politicians and administrators. The "science as usual" position met opposition by those who claimed that it seems incoherent to say on the one hand that the PP is directly linked to the state of knowledge,

i.e. the uncertainty of information, that science provides, while on the other hand leaves the burden of interpreting the significance of the incomplete state of knowledge to others who may lack the expertise to understand the uncertainties or see them in their appropriate context. To further stress the relevance to science, it is pointed out that the image of science as a linear accumulation of facts and the gradual eradication of all uncertainty is misguided. Uncertainty is increasingly seen as inherent to the production of scientific knowledge and may increase as knowledge increases. This is particularly so when our knowledge depicts unbounded complex or chaotic systems in nature as opposed to the idealised and controlled conditions of science in the laboratory. These systems are a challenge to the assumed ability of science to control and predict outcomes. It is furthermore claimed that risk assessments as practiced in regulatory science is strongly influenced by value decisions and non-scientific considerations. Thus, there is an intimate linkage between science and politics that seems to bespeak that the PP affects both the production of relevant scientific knowledge and the decision-making based on it.

2. Obviously, the PP has an important *legal* context. There is discussion whether precautionary action should be framed within a context of recognising an environmental law "principle", or whether one should rather talk about a precautionary approach when dealing with uncertain risks. The latter seems less demanding and open to alternative approaches as well. It seems a matter of fact that even states that strongly oppose the PP, have implemented policies in certain areas that are precautionary. Thus not having a generally binding legal principle still leaves room for precautionary action should a state decide so. The crucial question seems to be whether precaution has become part of customary international law. One element of the debate is the question of burden of proof. The invocation of the PP often requires either to shift some of the burden of proof showing the technology to be safe to those who develop and market the technology, or to relax somehow the standards of evidence for the suspicion of unacceptable risks (de Sadeleer 2002, Andorno 2004).

3. There is an important *political context* behind these issues as well. This can perhaps best be illustrated by pointing to the fact that acceptance or rejection of the PP is seldom coherent even within the domestic policies of a country, but seem to follow considerations of national interest. For instance, the USA has policies that are strongly precautionary in wildlife protection, but opposes the PP in a global trade context. Australia has domestic obligations to apply the PP in their national environmental policy decisions, but joins the USA in their resistance to accepting PP as an international legal principle. In other areas, e.g. within the Convention on International Trade in Endangered Species of Wild Fauna and Flora (CITES) or within the International Whaling Commission both countries are supportive of the PP. Within the EU one has noted that Southern European countries allow the sales of unpasteurised cheeses in spite of the risk that it may harbour *Listeria monocytogenes* and other dangerous bacteria. In this they seem to contradict the precautionary policies for food safety the EU propagates in other areas. They do so because of the long traditions of this kind of cheese making and their role in the food-culture of the countries. Such variation in the preferred approach to the PP within different areas of application easily gives rise to the suspicion that states support the PP when it can meet their environmental and other safety standards at little or no cost, but that they reject other states' use of it when this implies high costs for their own economy. In the context of globalisation of trade and technology it emerges that the interests of states to protect certain rights (IPRs) over a technology or the interest to export technologies to countries with less stringent safety regulations may further intensify the inequalities between the developing countries and the industrialised countries.
4. Finally, there is an *ethical and cultural* context. Our dealings with nature, our considerations of human health and our dealings with risks imposed on us by others are typically deeply embedded in a cultural framework of understanding and valuation. How risk-aversive or risk-taking people are in various areas is influenced by value-laden concepts and their role in the respective culture. Other values, e.g. values stressing individual autonomy versus values

conducive to social coherence, vary culturally. The same holds for religious versus secular values. The European/World Values Surveys provide evidence based on empirical data from almost 80 societies worldwide that post-industrial change brings remarkable changes in people's world-views (Inglehart 1997; Inglehart and Baker 2000; Inglehart and Welzel forthcoming). As the knowledge economy replaces the prominence of the industrial sector, values that emphasise conformity to group discipline and institutional authority tend to give way to values that emphasise human self-expression and individual choice (Welzel 2003). These attitudes have a profound impact on our views on moral responsibility. This applies e.g. to conceptions of both inter-generational and intra-generational justice. These cultural factors also have a large impact on how we view the moral standing of nature and wildlife.

One may roughly distinguish between a precautionary approach and the PP. This is relevant when describing the history. Precautionary "thinking" has been with humanity probably for a very long time and one may trace examples of it in the history of technology. Precautionary approaches also go back in history for quite some time. An important study on *Late lessons from early warnings* (Harremoës *et al.* 2001) mentions the example of Dr John Snow, who in 1854 recommended removing the handle of a London water pump in order to stop a cholera epidemic. The evidence for the causal link between the spread of cholera and contact with the water pump was weak and not a "proof beyond reasonable doubt". The simple and relatively inexpensive measure was very effective. The PP, however, seems of a more recent historical date, and it implies a comprehensive and legally binding obligation to use precaution in special cases.

History: The "Vorsorgenprinzip" in German environmental policy

The PP is one among altogether five central principles in German environmental policy (see Boehmer-Christiansen's contribution in O'Riordan & Cameron 1994.) The other principles are "the polluter pays", "cooperation" (*Kooperation*), "proportionality between costs and profit (*Wirtschaftliche Vertretbarkeit*) and "joint responsibility" (*Gemeinlastprinzip*). While the principle of proportionality indicates that no enterprise or trade should be subjected to higher costs than it is able to bear without going bankrupt, common responsibility means that any enterprise or trade can be subsidised in order to introduce measures to stimulate the environment. The PP may be traced back to the first draft of a Bill in 1970 aiming at securing clean air. This document expressed that the Bill aimed at preventing damaging environmental effects: the greater the danger, the greater the need for measures taken by the authorities to protect the people. This also set the legal framework for active measures that were not aiming at repairing damage that had already taken place. The law was passed in 1974 (as *Bundes-Immissionsschutzgesetz, BimSchG*) and covered all potential sources of "air pollution, noise, vibrations and similar processes".

The most unambiguous explanation and definition of the PP in German environmental policy came in a report from the Ministry of the Interior of the Federal Parliament (*Bundestag*) in 1984. Here it was stated that: "Responsibility towards future generations commands that the natural foundations of life are preserved and that irreversible types of damage, such as the decline of forests, must be avoided". Thus:
"The principle of precaution commands that the damages done to the natural world (which surrounds us all) should be avoided *in advance* and in accordance with opportunity and possibility. *Vorsorge* further means the early detection of dangers to health and environment by comprehensive, synchronised (harmonised) research, in particular about cause and effect relationships ..., it also means acting when conclusively ascertained understanding by science is not yet available. Precaution means to develop, in all sectors of the economy, technological processes that significantly reduce environmental burdens, especially those brought about by the introduction of harmful substances" (Bundesministerium des Innern, Dritter Immissionsschutzbericht, 1984, Drucksache Bonn 10/1345, p. 53; here quoted after the translation by Sonja Boehmer-Christiansen in O'Riordan, T. & J. Cameron 1994).

The combination of the PP with the development of cleaner technologies is typical of the German ideas of environmental protection. By way of structural measures one has given support to the development of technical solutions to environmental problems. In Germany the environment is first of all protected via the use of technology (BAT, "best available technology", *bester Stand der Technik* respectively). This has created jobs and environmental technology has become a growth area.

Defining the Precautionary Principle

The German interpretation of the PP is one of many definitions. There seems to have been little convergence yet towards a common definition of the PP in the various international treaties. The North Sea Treaties (Bremen 1984, London 1987, Den Haag 1990, Esbjerg 1995; all reprinted in Esbjerg 1995) are early examples of international treaties where the PP has had a very strong position. What is interesting is the shift of reference to the PP in the various North Sea Treaties:

From: "... timely preventive measures ..." given "insufficient state of knowledge" (1984) to: "... a precautionary approach is necessary which may require action ... even before a causal link has been established by absolutely clear scientific evidence ..." (1987) and: "... apply the precautionary principle ... even when there is no scientific evidence to prove a causal link ..." (1990) to finally: "... the guiding principle ... is the precautionary principle ... — ... the goal of reducing discharges and emissions ... with the aim of their elimination" (1995).

Scientists often criticise the notion of precaution as being too imprecise; that there is no definition available that allows an immediate operationalisation of the principle (cf. Sandin 1999; Graham 2001; Goklany 2001; Morris 2000). This is, of course, true for all the diverse definitions and formulations that this principle has undergone over the years. None of these formulations allow for a mechanical application of the principle. All need interpretation. The scepticism seems to persist in many quarters of science, in spite of the many academic efforts to clarify precaution further (cf. e.g. O'Riordan & Cameron 1994; FoS 1997, JoRR 2001, JAGE 2002; Cottam *et al.* 2000; Freestone & Hey 1996; Fjelland 2002; Raffensperger & Tickner 1999; Tickner 2003; see also Lemons & Brown 1995; Lemons 1996).

Here is the formulation that is the most cited in the literature on the PP:

Rio Declaration 1992, § 15:

"In order to protect the environment, the precautionary approach shall be widely applied by States according to their capabilities. Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."

There are several weaknesses in this attempt to define the PP. The Rio Declaration for instance can be criticised for trying to characterise the PP by using a ***triple*** negation ("... ***lack*** *of full scientific certainty shall* ***not*** *be used as a reason. ... for postponing cost-effective measures* [= ***not*** *acting*]" my emphasis). Many people have claimed that such a "definition" does not amount to operationalising the PP and that it remains inherently vague.

A recent UNESCO report under the auspices of its *World Commission on the Ethics of Scientific Knowledge and Technology* (COMEST) compares some of the better known versions of the principle (UNESCO/ COMEST 2005). In the following table we add some additional ones:

Source	Definition	Optional/Mandatory action
United Nations World Charter for Nature (1982)	*"[When] potential adverse effects [of activities] are not fully understood, the activities should not proceed."*	Strong: requires a moratorium in the case of uncertainty.
London Declaration (Second International Conference on the Protection of the North Sea 1987)	*"Accepting that, in order to protect the North Sea from possibly damaging effects of the most dangerous substances, a precautionary approach is necessary which may require action to control inputs of such substances even before a causal link has been established by absolutely clear scientific evidence."*	Weak: includes qualifying language such as "may require action" and "before ... absolutely clear ... evidence."
Rio Declaration (United Nations 1992b)	*"In order to protect the environment, the precautionary approach shall be widely applied by States according to their capabilities. Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."*	Weak: includes qualifying language such as "according to their capabilities" and "...postponing cost-effective measures." Contains triple negation.

contd.

International Joint Commission (1994)	*"All persistent toxic substances are dangerous to the environment, deleterious to the human condition, and can no longer be tolerated in the ecosystem, whether or not unassailable scientific proof of acute or chronic damage is universally accepted."*	Strong: bans use despite uncertainty of effects.
EU communication on the PP, 2000	*"The precautionary principle applies where scientific evidence is insufficient, inconclusive or uncertain and preliminary scientific evaluation indicates that there are reasonable grounds for concern that the potentially dangerous effects on the environment, human, animal or plant health may be inconsistent with the high level of protection chosen by the EU"*	Strong: requires intervention to maintain the high level of protection chosen by the EU.
Wingspread Statement on the Precautionary Principle	*"When an activity raises threats of harm to human health or the environment, precautionary measures should be taken even if cause and effect relationships are not fully established scientifically ... [The] proponent of the activity, rather than the public, should bear the burden of proof."*	Strong: clearly places the burden of proof on the proponent of an action to show that it does not pose a danger of environmental harm.

Already in 1994 it was pointed out (O'Riordan & Cameron 1994) that the vagueness of the principle is by no means surprising, nor is it a drawback. In 1999 Jordan and O'Riordan stated that "the application of precaution will remain politically potent so long as it continues to be tantalisingly ill-defined and imperfectly translatable into codes of conduct, while capturing the emotions of misgivings and guilt" (Jordan & O'Riordan 1999). The PP has a similar semantic status to moral norms or ethical principles (like human dignity, equity, and justice) or the principles of human rights. It needs to be interpreted and specified on a case-by-case basis, and it will sometimes change its specific content according to the available information and current practices. With ethical principles it is well recognised that for instance the protection of human dignity sometimes calls for a certain measure of paternalism (e.g. when institutionalising certain patients) while paternalism in other cases might be the direct opposite of respect for human dignity. This is quite similar to precaution. In order to protect for instance the biodiversity of a given region it may be a wise measure simply to leave a disturbed or polluted river leading into this region to its further natural course, and stop all kinds of human interaction with the river. But in some cases it may rather be indicated to take active steps to bring this river back into a quasi-natural state again, e.g. by restocking fish species, reducing its salinity etc. We need to look at the case at hand in order to find out what precaution means in that specific case. Partly this is due to the complexity of the scientific facts that we need to relate to. But partly this is also due to the varying interests and values that enter such a case. Typically there will be competing interests (aside from e.g. biodiversity) at stake, and sometimes these interests indeed deserve special attention (e.g. to preserve some cultural diversity by providing the economic basis for some human settlements). While the PP can remind us of our moral duty to prevent harm in general, it cannot prescribe what kind of sacrifice we should be prepared to make in each and every case. Thus the PP has the semantic status of a general norm rather than that of a detailed step-by-step rule of operation. It follows from this that it may make its occurrence in the guise of a multitude of different formulations and goal expressions.

Despite the differences in the wording, there are several key elements that most definitions or mentions of the PP in treaties have in common. These are, according to (UNESCO/COMEST 2005):

- "The PP applies when there exist considerable scientific uncertainties about causality, magnitude, probability, and nature of harm;
- Some form of *scientific analysis* is mandatory; a mere fantasy or crude speculation is not enough to trigger the PP. Grounds for concern that can trigger the PP are limited to those concerns that are *plausible* or scientifically tenable (that is, not easily refuted);
- Because the PP deals with risks with poorly known outcomes and poorly known probability, the unquantified *possibility* is sufficient to trigger the consideration of the PP. This distinguishes the PP from the prevention principle: if one does have a credible ground

for quantifying probabilities, then the prevention principle applies instead. In that case, risks can be managed by, for instance, agreeing on an acceptable risk level for the activity and putting enough measures in place to keep the risk below that level;

- Application of the PP is limited to those hazards that are *unacceptable*; although several definitions are more specific: Possible effects that threaten the lives of future generations or other groups of people (for example inhabitants of other countries) should be explicitly considered. Some formulations refer to 'damage or harmful effects', some to 'serious' harm, others to 'serious and irreversible damage', and still others to 'global, irreversible and trans-generational damage'. What these different clauses have in common is that they contain value-laden language and thus express a moral judgment about acceptability of the harm;
- Interventions are required before possible harm occurs, or before certainty about such harm can be achieved (that is, a wait-and-see-strategy is excluded);
- Interventions should be proportional to the chosen level of protection and the magnitude of possible harm. Some definitions call for 'cost-effective measures' or make some other reference to costs, while others speak only of prevention of environmental damage. Costs are only one consideration in assessing proportionality. Risk can rarely be reduced to zero. A total ban may not be a proportional response to a potential risk in all cases. However, in certain cases, it is the sole possible response to a given risk;
- There is a *repertoire of interventions* available:
  (1) measures that *constrain the possibility of the harm*;
  (2) measures that *contain the harm,* that is limit the scope of the harm and increase the controllability of the harm, should it occur;
- There is a need for ongoing systematic empirical search for more evidence and better understanding (long-term monitoring and learning) in order to realize any potential for moving a situation beyond the PP towards more traditional risk management" (UNESCO/COMEST 2005).

It was on the basis of these common elements that the working group that wrote the above mentioned report suggested a new working definition of the PP. The suggested definition is this:

Precautionary Principle, a working definition

When human activities may lead to morally unacceptable harm that is scientifically plausible but uncertain, actions shall be taken to avoid or diminish that harm.

Morally unacceptable harm refers to harm to humans or the environment that is

- threatening to human life or health, or
- serious and effectively irreversible, or
- inequitable to present or future generations, or
- imposed without adequate consideration of the human rights of those affected.

The judgment of *plausibility* should be grounded in scientific analysis. Analysis should be ongoing so that chosen actions are subject to review.

Uncertainty may apply to, but need not be limited to, causality or the bounds of the possible harm.

Actions are interventions that are undertaken before harm occurs that seek to avoid or diminish the harm. Actions should be chosen that are proportional to the seriousness of the potential harm, with consideration of their positive and negative consequences, and with an assessment of the moral implications of both action and inaction. The choice of action should be the result of a participatory process.

When to apply the PP?

The basic condition for the application is the presence of major scientific uncertainty. Note that risk alone, if not accompanied by uncertainty, does not qualify one to apply the PP. It may for instance be the case that a reliable risk assessment of a certain product shows that there exists a very low probability for negative health effects for certain groups of the population, e.g. small children. In that case one does not need to employ the PP. A policy of prevention may be sufficient, and one may e.g. decide that even such a low risk may be too high for the group in question. This is certainly dependent on one's values and the level of protection that a society tries to uphold. Yet, all this can be achieved without any recourse to the PP. Prevention is not the same as precaution.

The conditions for applying the PP can be spelled out in some detail. The conditions the Norwegian National Committee for Research Ethics in Science and Technology NENT (1997) adopted are essentially the following:

1. there exist considerable scientific uncertainties;
2. there exist scenarios (or models) of possible harm that are scientifically plausible (i.e. based on some scientifically acceptable reasoning);
3. uncertainties cannot be reduced without at the same time increasing ignorance of other relevant factors; (i.e. attempts to reduce uncertainties by e.g. model-building or laboratory studies typically imply abstractions that lead away from the real system under study and there is no "adding back" to real conditions; cf. Fjelland 2002)
4. the potential harm is sufficiently serious or even irreversible for present or future generations;
5. if one delays action now, effective counter-action later will be made more difficult.

While the NENT conditions for the application of the PP do not in any sense lay claim to expressing a widespread agreement, it is noteworthy that e.g. the EU communication on the PP (EU 2000) seems in part to express a similar spirit, for instance when it states that "recourse to the precautionary principle presupposes that potentially dangerous effects deriving from a phenomenon, product or process have been identified, and that scientific evaluation does not allow the risk to be determined with sufficient certainty".

It should be noted that all of these conditions need to be met. Without for instance the last condition being fulfilled one does not need to apply the PP. In such cases one may rather adopt a wait-and-see strategy.

Choice of precautionary strategies

Once one has established that the PP has to be applied, one faces the question of what to do about it. How precisely shall we act (including refraining from acting at all)? What measures should be counted as precautionary in some sense? This is the important question one has to address once the above conditions for the application of the PP are met. It is normally at this point that differences of opinion loom large.

Any action that can be assumed to effectively reduce the risk of the potential harm occurring, or that may contain the scope of the harm should it occur and that prepares us for handling the potential harm could be counted as a precautionary strategy. Given such a characterisation of a precautionary strategy, it seems clear that in most cases we have to select among a whole range of precautionary options. Choosing a strategy invariably involves taking a stand on basic value issues.

The EU Communication on the PP (2000) specifies a number of constraints on possible PP measures:

- ➢ non-discrimination (between identical problems in different areas)
- ➢ consistency (of policies)
- ➢ cost-benefit analysis (needs to be considered for action and non-action)
- ➢ proportionality (of measures in relation to possible harm)
- ➢ examination of scientific development (even after implementation)
- ➢ burden of proof (on those who propose a practice).

In a previous paper (Kaiser 1997) I argued that once it has been established that the PP should be applied, one is still facing a multitude of possible precautionary strategies. There is no one best strategy in any objective sense. One has to make trade-offs, for example between effects on nature and effects on society. This is certainly legitimate, but it is not a question of straightforward science. It is a value decision.

The example of xenotransplantation

It is, I think, useful to look at a specific example in order to see how the PP works or would work in practice.

Xenotransplantation is the transplantation of organs or body-cells from animals to human beings, for instance the heart of a pig. Xenotransplantation marks a qualitatively new challenge in medical technology assessment. The reasons for this claim are twofold: (i) in contrast to more traditional medical interventions, xenotransplantation involve risks not only to the patient, but also to larger segments of society, thus to public health in general; (ii) while most medical technologies demand assessment and risk-management at the time when the technology is sufficiently developed to be put into practice, xenotransplantation demands pro-active action at a very early stage of development.

The main risks of xenotransplantation stem from the possible harm that infectious diseases are transferred from animals to humans. Scientists identified the so called 'porcine endogene retrovirus' (PERV) as a possible infection of particular concern. To date no studies have demonstrated any direct transfer of PERV outside the laboratory from pig cells to human cells. But the scientists tend to agree that seven steps are necessary for PERV-infections to be a health risk to human populations:

1) PERV must be present in pig cells from the donor animal,
2) infectious PERV must be able to infect human cells,
3) PERV must be released from the transplanted organ or cells,
4) released PERV must be able to infect human tissue of the recipient,
5) PERV must be able to reproduce in the recipient,
6) PERV must be excreted and transferred to other humans, and
7) the PERV infection must lead to disease in humans.

Condition 1) and 2) were shown to hold in laboratory studies; conditions 3) and 4) were demonstrated in immune-deficient mice; the three last conditions could not yet be demonstrated. The fact that the possibility of each step is uncertain but scientifically plausible (no step can be ruled out), and that four of the seven steps necessary for the harm to occur were already shown to occur in laboratory studies, provides ground for concern. PERV is only one type of virus. There could be other viruses of concern that are not yet identified.

Further ground for concern arises from the scientific theory of zoonosis, which is widely known as one of the theories used to explain the origin of the HIV virus. According to this theory, HIV-infections have developed by zoonosis: viruses from apes became able to reproduce themselves in the human body after some initial contact with the animal, and were then spread to other humans through human contact.

Given these considerations one might conclude that:

a) there exist significant scientific uncertainties about the possible infectious consequences of xenotransplantation,
b) there exist scientifically-based models that indicate a possible scenario of harm (zoonosis),

c) this harm could be potentially great and difficult to contain and might be irreversible,
d) the harm affects an important value: human health,
e) once infectious diseases are transferred it may be too late to do something about it, and
f) there is no scientific proof that xenotransplantation can cause new viruses for humans, but
g) it is not feasible to reduce the uncertainties significantly without at the same time increasing the risk that the harm might occur, that is, perform xenotransplantations.

Conditions a)–g) can be seen as general conditions for applying the PP. Thus, precautionary measures might be indicated in this case.

Using the new definition of the PP provided by COMEST, one may also note the following: Xenotransplantation might lead to morally unacceptable harm, since human (population) health/life is potentially at stake. The evidence cited to show significant uncertainties is based on plausible scientific considerations, and not on mere speculation alone. There is significant uncertainty both in respect to what exactly might cause the potential harm, and in respect to the scope of that possible harm. A number of actions seem possible to either prevent the envisaged harm or to restrict it should it occur. This is discussed in the following paragraph.

What then are the precautionary strategies that one might want to implement as a consequence? A precautionary strategy can be defined as any measure that can be believed to effectively reduce either the risk of the harm itself, or the magnitude and spreading of the harm, should it occur. A Norwegian Governmental Commission Report (NOU 2001) discusses a number of possible strategies: a moratorium, a step-by-step and a case-by-case strategy, restrictions of uses to small and strictly monitored groups, and the international cooperation in monitoring the patients (and their families). The first is the strictest and the last is the most liberal, i.e. least effective strategy. As tempting as a moratorium may look from a societal point of view, it should be kept in mind that it only delays the problem. It might actually backfire, given that not all countries might implement a moratorium and that diseases know no borders. What one eventually wants to achieve is enough knowledge and a strong institutional apparatus to contain the possible harm should it materialise, but still allowing the technology to develop for the benefits of patients. However, it is clear that any decision between these different precautionary strategies will be strongly influenced by value-assumptions and rest in the final instance on political decisions.

Conclusion

The Precautionary Principle has triggered extensive debate both among scientists and in political circles. The focus on scientific uncertainty and the need to manage uncertainties represents a major regime change in the way science serves as the provider of premisses/information for environmental and health policy. The PP demands that the scientist spells out all the relevant uncertainties that pertain to a situation. Furthermore, the scientist needs to assess whether there exists some scientifically plausible evidence or some science-based model that would indicate a scenario of possible future harm. This exercise asks the scientist to leave the dominating strong standards of proof within science behind, and use qualitative judgement in screening scientific knowledge for indications of what a certain technology, intervention or practice may lead to. The scientist must be prepared to engage in extra-scientific platforms with decision makers, stakeholders and the general public. Here the scientist should be ready to focus on values that are at stake and how science can contribute to protect human health, safety and the environment. Science is challenged to come up with a variety of possible precautionary strategies if the PP is to be employed, and to discuss them critically in their relevant context. The close relation to value aspects and ethics, bringing value aspects to the surface, is a challenge that scientists may not be quite prepared for yet. On the other hand, it may be precisely because of these aspects that the PP enjoys a large support in wide circles of the European population. It represents a novel idea of how scientific knowledge may indeed contribute to progress. Progress is, after all, a value concept.

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On Re-staging the Universal: Butler, Hegel and Contesting the Closure of Logic[1]

Valerie Kerruish

1. Preliminaries

"Formalism", Judith Butler writes,

> is not a method that comes from nowhere and is variously applied to concrete situations or illustrated through specific examples. On the contrary, formalism is itself a product of abstraction, and this abstraction requires its separation from the concrete, one that leaves the trace or remainder of this separation in the very working of abstraction itself. In other words, abstraction cannot remain rigorously abstract without exhibiting something of what it must exclude in order to constitute itself as abstraction.[2]

The general context of Butler's essay 'Re-staging the Universal' is a consideration of universality in the political realm which, while fully apprised of the 'false universality' of colonial and imperial projects,[3] wants to restage the universal as a project of cultural translation.

In the essay Butler reads Hegel as calling into question whether formalisms are ever really as formal as they purport to be.[4] Reading paragraphs 19 to 25 of Hegel's *Encyclopaedia Logic* she presents Hegel's approach to universality as proceeding by way of successive revisions of the notion of universality. Thus the form (product) and character of thought are a) universal qua 'abstract'. But then thinking as activity yields b) the

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[2] Judith Butler, 'Re-staging the Universal' in Judith Butler, Ernesto Laclau and Slavoj Žižek, *Contingency, Hegemony, Universality: Contemporary Dialogues on the Left*, London, Verso, 2000 at 19.

[3] Butler, above n.2 at 15.

[4] Butler, above n.2 at 14–15.

active universal which produces c) a deed as the universal. Thus three different names are offered for a universality, both singular and various, to which is added d) the subject, the pronomial 'I' as also the universal. Hegel, she argues, is inhabiting a Kantian voice prior to a critique of Kant for suppressing the internal form of d) — the external form being, here, communality. Taken abstractly, the 'I' is pure relation to itself "in which abstraction is made from representation and sensation, from every state as well as from every peculiarity of nature, talent, experience".[5] Such a positing of the universal 'I' requires the exclusion of what is specific and living from the self and since this too is universal we get a doubling designated in terms of abstract and concrete. In general, Hegel's point against Kant, according to Butler, is made by showing in various contexts that

> when the universal is conceived as a feature of thought, it is by definition separated from the world it seeks to know.[6]

To the extent that freedom of thought guarantees freedom, freedom is defined precisely over and against all exterior influence and this abstract freedom intrinsic to thought, brings a certain hubris, or will to mastery, to be countered by 'humility', 'modesty' that is attained by immersion in the *matter* itself. "Hegel will conclude", Butler writes,

> that not only is the thinking self fundamentally related to what it seeks to know, but the formal self loses its 'formalism' once it is understood that the production and exclusion of the 'concrete' is a necessary precondition for the fabrication of the formal. Conversely, the concrete cannot be 'had' on its own, and it is equally vain to disavow the act of cognition that delivers the concrete to the human mind as an object of knowledge.[7]

Butler will go on to draw from this her counter to Bataille's and Derrida's dubbing of Hegel's thought as the thinking of mastery (as distinct from sovereignty),[8] via a consideration of Hegel's phenomenological (i.e. in *The Phenomenology of Spirit*) linking of universality to reciprocal

[5] Ibid at 16.

[6] Ibid at 17.

[7] Ibid at 18.

[8] Ibid at 19; see further J. Derrida, 'From Restricted to General Economy A Hegelianism without Reserve' in *Writing and Difference*, Alan Bass trans., London, Routledge, 1978, 251–277; Joseph C. Flay, 'Hegel, Derrida and Bataille's Laughter' and Judith Butler 'Commentary on Joseph Flay' in *Hegel and his Critics: Philosophy*

recognition and the role of customary practice or *Sittlichkeit* as a substantive rather than formal condition of recognition. From the implicit rejection of transcultural norms in this thinking Butler moves to the performative, cultural translation, as a possible forging of universality which crosses cultures without transcending culture.

2. Dilemma

This paper began its life, some years ago, as an attempt to supplement Butler's essay by distinguishing logical formalisms from the theoretical formalisms of which she is critical. Formal thought, I claimed, does remain rigorously itself, as a practice of mathematical logic, quite simply by adhering to the assumptions, definitions, axioms (if any) and rules it has set for itself. If even so it runs into antinomies and comes up with paradoxical results these should be seen to inhere in something other than the contamination of the abstract by the concrete in the formalisms of which Butler is critical. That something other is the role of contradictions in concept formation. What I was aiming at, on the basis of my own reading of Hegel's *Logic*, was the possibility of turning his idea of a dialectic of pure reason to the task of questioning all forms of authority, including that which may be thought to inhere in classical logic itself.

That remains my aim, but it now seems to me that a distinction between 'logical' and 'theoretical' formalisms made in terms of formal thought 'remaining rigorously itself' is inapposite. The 'I' of a Kantian consciousness, as it clings to Kant's transcendental unity of apperception,[9] is very much in question in Hegel's critique of the critical philosophy. His attempt to replace Kant's foundational notion of the transcendental unity of apperception with his own logical foundation and so overcome Kantian and Fichtean 'I's is the undertaking of the *Logic*. Now this, I think, is

in the Aftermath of Hegel, ed. W. Desmond, Albany, State University of New York Press, 1989, 163–178.

[9] "It is one of the profoundest and truest insights to be found in the *Critique of Pure Reason* that the *unity* which constitutes the nature of the *Notion* is recognised as the *original synthetic* unity of *apperception*." G.W.F. Hegel, *The Science of Logic* [1816], trans. A.V. Miller, Atlantic Highlands, New Jersey, Humanities Press International Inc., 1969 at 584; *Wissenschaft der Logik*, Zweiter Teil, *Sämtliche Werke* v.5, Jubiläumsausgabe 4th ed., Stuttgart-Bad Cannstatt, Friedrich Frommann Verlag, 1964 at 15.

what Butler neglects, as I shall argue below. But there is a further problem affecting my own argument. Things have moved on very considerably in the field of formal logic since Hegel's day. If it is now proposed that philosophy and more particularly political philosophy should allow that the logic of any such logical foundation should be presentable, in conformity with that development, as a formal system of mathematical logic, a problem of discipline and practice cannot be ignored.

However interdisciplinary the exchange between formal logic, mathematics and philosophy was in the first decades of the last century, the subsequent development of mathematical logic has been as a mathematical discipline. Such relations as have been maintained with philosophy, whether designated 'philosophical logic' or 'the philosophy of logic(s)' have been very largely in the very tradition of formal logic which Hegel rejected. The situation currently inherited is one in which it can be and is argued that no formal system that is not 'complete' (meaning roughly fully formalisable) should be permitted the designation of (formal) 'logic'. 'Logic' in this view is confined to classical propositional logic and first order predicate logic.[10] This position in debate in philosophy of logic concerning the nature of logic is no doubt conservative, but it sets a parameter of that debate which could be taken as the 'precise' meaning of formal thought 'remaining rigorously itself'. This was and is certainly not the meaning of thought 'remaining rigorously itself' that I intended. On the other hand, if the practice of doing mathematical logic, which standardly includes constructing or working within incomplete systems of higher order logic and set theory, is taken as the practice in which formal thought 'remains rigorously itself' it must be allowed that this is a practice of mathematics.

The stumbling point here is not that it is, as such, not philosophy, but the interaction (or lack of it) between mathematical logic and philosophy. Such interaction would take the form of foundational research but as far as I can see disciplinarity has extended into this field too. Foundations of mathematics remains a (small) field within mathematical logic which may be philosophically engaged. Such engagement however tends to be confined to a specialist philosophy of mathematics that is largely within the analytic tradition. Elsewhere philosophy, has tended toward critique

[10] Susan Haack, *Philosophy of Logics*, Cambridge, Cambridge University Press, 1978 at 6, discussing the view of W.C. Kneale.

or deconstruction of metaphysics or to hermeneutics and taken a variety of 'ethical turns' on questions of justification. Alain Badiou's *L'être et l'événement* (1988)[11] with its extended interpretation of ZFC set theory[12] as ontology is a notable exception. But this standard set theory is a classical theory which has guarded itself from paradox by a judicious choice of axioms.[13] As a form of higher order logic it is designed to avoid the contradictions which a dialectical and speculative logic in the spirit of Hegel seeks to accommodate.

'In the spirit of Hegel': can this be claimed? Compared to Descartes, to Spinoza, to Kant, Hegel turns philosophy's back to mathematics most emphatically, heaps contempt on Leibniz' "immature" idea of a symbolic universal language of thought and declares the German language preeminently suitable to his enterprise! Given his insistence on the inseparability of form and content, and on (his) dialectical method as the universal aspect of the form of the Notion,[14] to revise Hegel on this point can surely be said to reject his philosophy.

I am drawn two ways by such saying. One inclination is to say yes, but that is no obstacle to finding in Hegel's thought a questioning of authority and of law that I think comes from his idea of thought's (dialectical and speculative) logical foundation. Call the resulting discourse an interdisciplinary legal and political 'theory' rather than 'philosophy'. The name is irrelevant to an enterprise that will still, as a matter of its method, require attention to Hegel's texts and only differs from other interdisciplinary theories of law and politics in its reference to mathematical logic. The other is to ask, counterfactually: how would Hegel have responded to the antinomy that sank Frege's hopes of proving Kant wrong on the nature of arithmetic by reducing it to a formal mathematical logic? He might have been happily surprised to find his idea of dialectic as a necessary function

[11] Translated by Oliver Feltham as *Being and Event*, London, Continuum, 2005.

[12] Zermelo-Fraenkel set theory with the axiom of choice

[13] Badiou explicitly defends his commitment to classical logic (as against intuitionistic logic) in Meditation 24.

[14] I use the term 'Notion', following the English translations of Hegel used, to designate 'concept' (*der Begriff*), as distinct from 'idea' (*die Vorstellung*, often translated as 'figurate conception' or 'pictorial thinking' but better understood as a vague, imprecise idea) and 'Idea' (*die Idee*, a realised Notion or a Notion that is adequate to its content and so, in Hegel's sense "the objective truth or the truth as such") (Hegel, above n.9 at 755; 236).

of reason gain support from an unexpected quarter and revised his own judgement on the means and methods appropriate to accomplishing the aim of his logic.

Between these two responses, the motivation that first prompted my engagement with Butler's essay, still presses. I would like to see political thought 'on the left' engaging with a discipline born of an originally inter-disciplinary exchange between formal logic, mathematics and philosophy which has played no small part in the development of the machines which are employed to write, publish and circulate their ideas. No doubt, as Hegel somewhere remarks, one does not have to study the digestive system in order to digest one's food. No more can a demand be placed on political theorists to study recursion theory (or the theory of computability) in order to read and write texts, produced with the aid of a word processor, on articulations of power within a social order. Still, where abstraction and formalisms arising from it are topics in a project of restaging the universal, I see disciplinary and practical barriers but no justification for setting the universality of concepts in mathematical logic beyond the horizon of engagements. I would go so far as to suggest a lordly contempt of slaves, tools and machines within this attitude.

Re-staging the universal, as that is called for in left political theory, does not, in my view, rest with the possibility of re-staging through cultural translation, although it needs that too. It does not rest there because, in my view, (formally, mathematically) logically constructed universals and the hegemony of classical logic should fall within the challenges taken up by such theory. No doubt legal and political theory is far removed from the logical realm. That is to say, the concepts deployed in and more or less systematically organising such theory are both multiply mediated and separated by gaps from concepts, perhaps indicated by the same word (for example, 'reasonable', 'necessity', 'freedom'), which are located in the logical realm. But if Hegel's idea of thought's logical foundation is being taken up, then a question of how and the extent to which pure reason's forms and functions are relevant to such theory arises. It is a question that goes, in Hegel's terms, to how reason and the reasonable are conceived.

If it is being taken up: that is *my* enterprise and it lurches straight back into the stumbling point mentioned. It is *not* what is being taken up by Butler and the theorists with whom she is in dialogue, Laclau and

Žižek. Quite to the contrary, Laclau regards Hegel's philosophy as 'panlogicism' and while Butler questions that, her arguments for the openness of Hegel's dialectic rest on the impossibility of a purely formal discourse. Pursuing the dictate of the subjective and motivating 'should' of the previous paragraph brings me into conflict with her too. Evidently enough, the difficulty stems from exclusion of a notion of 'the formal' that is applicable to contemporary formal, mathematical logic from the ambit of her claims. But this exclusion disables the argument which I wish to make, namely, that Hegel while rejecting the sense of 'the formal' which separates form from content, *does* intend his *Logic* as a logic of pure thought; as 'the formal' of and in his way of thinking or 'the formal' of pure reason.[15] It seems then that the supplement I envisaged, as called for in political theory, cannot do what I wanted it to do, that is, leave Butler's ideas for re-staging the universal in place — the place being cultural theory — while using Hegel's idea of thought's logical foundation to give a critical standpoint *vis à vis* authority claims *without* distinguishing 'logical' and 'theoretical' formalisms.

A case then of damned if one does and damned if one doesn't? Indeed: a dilemma of the times.

3. Hegel: Thematically

> Philosophical thinking in general is still concerned with concrete objects — God, nature, spirit: but logic is concerned only and solely with these thoughts *as thoughts*, in their complete abstraction.[16]

To my mind, Hegel's thinking has both its radically emancipatory moment and its logical character in this aspiration. His idea for a dialectical and speculative logic is, as I read him, a foundational idea that pushes Kant's

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[15] "Logic being the science of the absolute form, this formal [science] (*dies Formelle*), in order to be *true*, must possess in its own self a *content* adequate to its form; (Hegel, above n.9 at 594; 29). Translation of *dies Formelle* as "this formal science" reads well but occludes the substantive of the German text, *das Formelle*, 'the formal'.

[16] G.W.F. Hegel, *The Science of Logic* [1812], trans. A.V. Miller, Atlantic Highlands, New Jersey, Humanities Press International Inc., 1969 at 34; *Wissenschaft der Logik*, Zweiter Teil, *Sämtliche Werke* v.4, Jubiläumsausgabe 4th ed., Stuttgart-Bad Cannstatt, Friedrich Frommann Verlag, 1964 at 24.

transcendental turn to a fully logical turn via arguments on two fronts. On the one hand, it takes issue with Kant for leaving formal logic outside the scope of his first *Critique* and so with the distinction between formal or general and transcendental logic in the critical philosophy. On the other hand, he takes issue with the irresolution of Kant's turn against the idea that thought is dependent for its content, albeit as mediated by the pure forms of intuition (time and space), on sensible objects.

Hegel's difference with Kant as regards the nature of 'logic' is formulated as the difference between regarding logic as a canon of judgement — in Kant's own terms, a priori principles of how the understanding ought to think[17] — and as an organon or tool for the production of objective insights. The argument is about the nature (and so the authority) of reason, that is, thinking in terms of relations between concepts, and it takes in reason's relation to the understanding, that is thinking in terms of bounded concepts. It is a basic and intractable difference.[18] The merely regulative role given to reason in Kant's philosophy is Hegel's abiding objection to it. The power of thinking the unconditioned, which for both is reason's claim, is denied access to 'truth' in its cognitive (theoretical or speculative) exercise in Kant's philosophy. Confined in this power to what may be learned in its pure practical exercise, reason's most valuable accomplishment, the reasonable, turns out to be an 'ought to be' which 'is' not in this world and is thus unknowable by reason in its cognitive moment.

Hegel states his *logos* preliminary to the body of derivations of the *Logic*,[19] as "the objectivity of illusion and the necessity of contradiction that belongs to the nature of thought determinations". It is, he says "nothing else but the inner negativity of the determinations as their self-moving soul, the principle of all natural and spiritual life." Placed into an homage to Kant for freeing dialectic from its reputation as arbitrariness

[17] "Logic is a science of reason not only as to mere form but also as to matter; a science a priori of the necessary laws of thinking, not, however, in respect of particular objects but all objects generatim; it is a science, therefore, of the right use of the understanding and of reason as such, not subjectively, i.e. not according to empirical (psychological) principles of how the understanding thinks, but objectively, i.e. according to a priori principles of how it ought to think." Immanuel Kant, *Logic* [1800], trans. Robert S. Hartman and Wolfgang Schwarz, New York, Dover Publications, 1974 at 18.

[18] See above p.30.

[19] For the distinction made here see below n.32

and showing it to be a "*necessary function of reason*",[20] and formulated as an extension of Kant's Antinomy of Pure Reason, this statement indicates Hegel's standpoint. He envisages a science of this *logos* — a logic in a classical philosophical sense — in which contradictions are relevant in a way that is both limitative (negative, dialectical) and constitutive (speculative).

I thus take Hegel at his word as intending his logic *as a logic* that will *replace* all previous metaphysics.[21] The first part, the Objective Logic with its purported derivation of the categories of the understanding, takes the place of ontology and ontotheology. The second part, the Subjective Logic, has the logical forms (of judgement, syllogism, theoretical and practical reasoning) as its subject matter. That is for Hegel the activity of the Notion which, as 'derived' in the final transition of the Objective Logic (so as the logical Notion or concept of concept) is *the form* of the conscious and self-conscious subject. Form and content are dialectically related in Hegel's thinking. They are not identical and nor are they independent of realm. The forms of consciousness and self-consciousness into which *Geist* or Spirit ('the Hegelian subject' or 'thinking subject' in Butler's terminology) enter, are logical forms, products of pure thought. In that sense of 'subject' that is always already embodied in the social relations of a place and time, Hegel's *Logic* is subjectless. In a metaphor that is not without its own interest in contexts of logic and politics, he writes of the Subjective Logic that its "task is to remodel an ancient city, solidly built, and maintained in continuous possession and occupation".[22] He is talking about the Aristotelian tradition of formal logic that he has elsewhere described as a "heap of dead bones" a "dull and spiritless reckoning": an incarnation *par excellence* of the formalism of which he was critical, but which is not the less part of the heritage.

Hegel certainly sticks to a resolve to make no use of much of *that* formal logic — and it might be added that the tradition returns the compliment by ignoring Hegel — but this is not to say that the logic he intends does not challenge it.[23] There is perhaps disinterest in, rather than a challenge to, the *correctness*, within its own frame, of the formal logic of

[20] Hegel, above n.16 at 56; 54.

[21] Ibid at 63f.; 64f.

[22] Hegel, above n.9 at 575; 3.

[23] In some interpretations, Hegel is read as making no challenge to "ordinary logic"; see e.g. Terry Pinkard, 'A Reply to David Duquette' in *Essays on Hegel's Logic*, ed.

his day. Rather, the frame itself, with its separation of form and content, is rejected. Hegel's 'logic' is framed as an encompassing unity within which form (the Notion, universality) and content ('truth' in Hegel's sense) are inseparable *and* for which the method is the universal aspect of the form. In Hegel's own terms the task undertaken in the Subjective Logic is "to go further [than the Aristotelian undertaking] and to ascertain both the systematic connection of these forms and their value".[24]

Given that Hegel has scant interest in and correspondingly scant specialist knowledge of the tradition of formal logic, the subsequent transformation of that tradition by mathematical logic is somewhat to the side of his thought. What that transformation brought about and continues to bring about, are new questions, questions that are addressed to classical logic. Its so called 'unquestionability', its status in Kant's eyes as a complete and perfect science, has become an anachronism. Further, updating to one of the seminal contributors to that transformation, the universality of (classical) logic as Frege (and Russell and Whitehead) conceived that,[25] came at the price of provable contradictions or antinomies when, in pursuit of the logicist attempt to reduce arithmetic to logic, certain axioms governing concepts were included. No doubt the import of the logical and set theoretical antinomies can be and standardly is pushed away and minimised, but that is not to say that this strategy is the best or even a good one. Indeed, explicit or implicit claims that this is the most 'reasonable' strategy, or that it serves 'us' and 'our science' best, are what I am opposing. Possible attitudes to contemporary scientific practices and institutions do not lie on a linear scale between unquestioning acceptance and horrified rejection. Questions of justification, community and objectivity in thought raised by Hegel's idea of thought's dialectical and speculative logical foundation considered from a perspective provided by current research in non-classical logics, open other possibilities. It is difficult to gain and communicate that perspective, but I do not accept the authority of those who declare it impossible or the judgement of those who think it unnecessary. Even as a matter of determining which logic is

G. di Giovanni, New York, State University of New York Press, 1990, 17–25 at 19f. The trouble is that what is meant by "ordinary logic" is entirely unclear.

[24] Hegel, above n.9 at 595; 31.

[25] Jean van Heijenoort, 'Logic as calculus and logic as language' *Synthese* 17 (1967), 324–330.

best suited for which scientific or technological endeavour the latter field of research works in the breach of the universality of classical logic.

So far as Hegel's idea of foundation and the questions raised by it are concerned, the relationship between the theory of knowledge or justification of *The Phenomenology of Spirit* and the poiesis and practice of the *Logic*, and the particular function that the *Logic* has within Hegel's system of philosophy are basic. In describing the 'result' of *The Phenomenology* as the presupposition of a presuppositionless logic, I take Hegel to be asking his readers to think twice about the relationship in question.[26] In my view, Hegel takes *The Phenomenology* to have shown that *its* presupposition or starting assumption, that according to Hyppolite of all theories of knowledge,[27] namely the distinction between subject and object (correlatively knowledge and being, in itself and for itself, certainty and knowledge) has been shown to be inadequate to science.[28] The standpoint of absolute knowledge absorbs this distinction in a double character of its own. It is a standpoint of a journeying consciousness which has experienced a range of attitudes to its desires and their reversal, and knows itself as the recollecting totality of that experience. It is also the abstract concept of pure science, of 'logic' in Hegel's sense, conceived but not yet realised by a derivation of the determinations of pure thought. The poiesis and practice of the *Logic* is the realisation of that concept *in the logical realm*: a realm that is a construction of thought and which contains no objects other than objects which thought gives to itself, namely concepts and operations on and with those concepts (judgement, syllogism, reasoning). Implicit, to my mind, is the thesis that it is *only* in the logical realm, only where thought by an artifice of idealisation, constructs a realm of absolute freedom, where it is, so to speak, alone with itself, that the ideal unity of theory and practice of knowledge is realisable.

The particular function of this part of the system is to provide thought's logical *foundation*, in a sense of 'foundation' that has its theory of justification in an epistemology and the activity of providing the foundation in logic. In this respect it is a departure from the *kind* of foundation that Kant sought. It presupposes the possibility of objective

[26] Hegel, above n.16 at 68; 71.

[27] Jean Hyppolite, *Genesis and Structure of Hegel's Phenomenology of Spirit* [1948], Evanston, Northwestern University Press, 1974 at 575.

[28] William Maker 'Beginning' in *Essays on Hegel's Logic*, above n.23 at 27–43.

knowledge (science) but not, as Kant supposed, as already instantiated in mathematics and theoretical physics. Rather, at the beginning of the *Logic*, it is a foundation that is yet to be provided, although it admits no doubt that it can be provided in the logical realm.

4. Hegel: Abstraction and Abstraction

There is a sense in which philosophy throughout Hegel's encyclopaedic system is 'logic', (a sense that leads Hao Wang to liken Hegel's conception of 'logic' to that of Wittgenstein in 'On Certainty'[29]). Philosophy, in Hegel's conception of it is throughout concerned with the Idea albeit in its different modes (*Weisen*) of existence, and throughout concerned with the 'derivation' of the categories for 'objective' knowing of a given realm. The logical mode of the absolute Idea is however its universal mode. Thought is concerned with itself in the *Logic*, with its own forms and functions, not with its modes of existence (*Dasein*) in the world, in the realms of nature and spirit.[30]

The transitions in Hegel's *Logic* are not and are not intended to be derivations in the sense of inferences drawn on the basis of rules of a formal logic, old or new. They are 'logical' in the sense that they purport to follow a movement of thought thinking thought by a method that takes nothing from outside the realm of pure thought other than the initial intuition of how, practically, that can be done. True enough, as Laclau says in his discussion of panlogicism,[31] the account of the method of the *Logic* comes only at the end, after the last transition (from the practical or objective Idea or Idea of the good, to the absolute Idea). The body of derivations which are the practice of 'doing logic' in Hegel's sense,[32] are however a *use* of the method. Evidently enough, some idea of how to perform the first 'derivation' (of Becoming from Being and Nothing) must precede it. Differently put: Hegel must have had some idea of how that 'complete

[29] Hao Wang 'What is Logic' *The Monist* 77 (1994), 261–277.

[30] Hegel, above n.9 at 824–5; 328.

[31] 'Identity and Hegemony' in *Contingency, Hegemony, Universality*, above n.2 at 61.

[32] Cf. Hans-Georg Gadamer, 'The Idea of Hegel's Logic' in *Hegel's Dialectic: Five Hermeneutical Studies*, trans. P. Christopher Smith, New Haven and London, Yale University Press, 1976 at 86, for properly hermeneutical comments on what "can properly be called Hegel's text".

abstraction' from the concrete objects of philosophical thinking was to be achieved.

This is the issue addressed in the chapter of the Greater Logic 'With what must science begin?'. Apparently ambiguous, its placement (after the Prefaces and Introductions; before the derivations) is the structural counterpart in that text to the final chapter which gives a reflective account of the method. And for all that I think Hegel judges badly (i.e. on the basis of prejudice and philosophical hubris) and wrongly (in terms of choosing the tool he needs for the end in view) in his lofty dismissal of Leibniz' ideas for using mathematical methods in the pursuit of logic, he is not without his insight here. Taking ordinary language as a given basis, Hegel takes predication in terms of the copula 'to be', and objectification as the turning of a propositional form (predicate or concept) into an object, as operations whereby thought is able to give an object to itself and thus to come up with Being ('is' turned to 'isness') as a meaningless object and beginning of his science.

> Whatever richer names be given to [the beginning of science] than is expressed by mere *being*, all that can be considered is *how such an absolute enters into the thinking cognition* [predication V.K.] *and into the expression of this cognition* [objectification V.K] (my emphasis).[33]

I do not claim that Hegel is seeing other than through a glass darkly. I do not deny that this interpretation is made from a perspective provided by contemporary mathematical logic with its use of formalised operations

[33] Above n.16 at 77; 83. Dieter Henrich in 'Formen der Negation in Hegels Logik' in *Seminar: Dialektik in der Philosophie Hegels*, ed. Rolf-Peter Horstmann, Frankfurt am Main, Suhrkamp, 1978, 213–229, draws attention to the '*substantivierte Aussageform'* (propositional form turned noun) as a basic operation in Hegel's logic, but he takes it as converting 'not' to 'nothing'. Yet Being is Hegel's first category and it is, so Hegel, pure thought's capacity to give itself an object that distinguishes philosophy from the other sciences (G.W.F. Hegel, *Logic: Part One of the Encyclopaedia of the Philosophical Sciences* (1830), trans. W. Wallace, Oxford, Clarendon Press, 1975, §17; *Sämtliche Werke* v.8, Jubiläumsausgabe 4th ed., Stuttgart-Bad Cannstatt, Friedrich Frommann Verlag, 1964. He also (in a *Zusatz*) speaks of making 'is' an object of investigation (Ibid §24Z at 40; 88): "... Being is a pure thought-determination: yet it never occurs to us to make 'is' (*das Ist*) an object of our investigation."

of predication and abstraction (objectification).[34] I do place quite some weight on this beginning. Structurally, and in terms of clear acknowledgement of *The Phenomenology of Spirit* as the presupposition of a presuppositionless logic,[35] this chapter sees Hegel *constructing* a realm of pure thought; finding the means whereby thought can make itself its own object; can turn itself back on itself. It is a capitalisation on the increased degree of reflexivity in Kantian philosophy that turns reflexivity to self-reference with the aim of establishing the universality of thought in its most extensive freedom. Seen again from the perspective of contemporary logic the antinomies and undecidability results of logical self-reference may be seen as vaguely and imprecisely anticipated.[36]

I am following through here on that aspect of the standpoint of absolute knowing that is Hegel's idea of and for his *Logic*. Objective knowing in Hegel's thought depends neither on the distinction between content and form characteristic of traditional formal logic nor on a process of abstraction appended to the ontological view of empiricist realism (the view that the material given by intuition and representation is real in contrast to the Notion) and its correlative characterisation of the empirical (as 'concrete') over the ideal (as 'abstract'). In Hegel's eyes,

> [i]n this view, [the view of empirical realism V.K.] to abstract means to select from the concrete object for *our subjective purposes this or that mark* without thereby detracting from the worth and status of the many other properties and features left out of account; on the contrary, these as *real* retain their validity completely unimpaired, only they are left yonder, on the other side; thus it is only the *inability* of the understanding to assimilate such wealth that compels it to content itself with the impoverished abstraction.[37]

[34] This interpretation of Hegel's beginning has been much discussed with Uwe Petersen whose non-classical symbolic logic 'in the spirit of Hegel' and whose explanations and guidance concerning that and other aspects of contemporary mathematical logic have enabled me to gain that perspective. For a presentation of that logic: Uwe Petersen, 'Logic Without Contraction as Based on Inclusion and Unrestricted Abstraction' *Studia Logica* 64, (2000), 365–403. For my own perspective see the Appendix to this paper, below.

[35] Ibid at 68; 71.

[36] See J.N. Findlay, above p.36.

[37] Hegel, above n.9 at 587; 20.

Hegel opposes this view with the idea of abstraction as a grasping of what is essential in the 'matter' with which thought is engaged be that, in the logical realm, itself, or in the realms of nature and spirit, the sensuous and sensuous-supersensuous material of appearance.

As regards Butler's arguments: the 'self' of that 'itself' (what she terms 'the formal self') in the *Logic*, and so as a 'derived' and in that sense justified concept, is the Notion in its logical form. It is neither reducible to method nor capable of being thought without a method: content, form and method are inseparable; their separation *is* 'formalism'. In so far as that method involves objectification ('abstraction' in the context of mathematical logic[38]) it is not abstraction *from* anything. It is constructive rather than reductive in character. Butler is quite right in moving back to *The Phenomenology* to link universality (as variously manifested in the logical Notion) to reciprocal recognition and the role of customary practice in presenting 'the thinking subject' (the Notion that thinks as situated, embodied, sociable human individuals). Hegel indicates this within his logical dialectic by incorporating into the *Logic* a section of *The Phenomenology* portraying the unresolved contradiction of self-consciousness.[39] What I think she neglects with her claim that formal thought cannot remain rigorously itself without displaying the contamination of the excluded concrete, is the extension of Kantian reflexivity to self-reference within a purely 'logical' realm. A 'doubling effect', paradoxes and antinomies that may attend on self-reference and are effects of self-reference rather than the contamination of an excluded 'concrete', is thus left out of her account of Hegel's logical thought.

In conceptual terms the loss is of the notion of 'formal' of which Hegel writes with reference to 'the formal' (*das Formelle*) of *his* logic.[40] It is a different conception of 'formal', or perhaps one should say it is a different way of thinking 'formal', from that which inhabits the tradition of formal logic. Now contemporary mathematical logic is no doubt a formulation of formal logic.[41] In that dimension, it retains and continues the tradition

[38] See Appendix, point 5, below.

[39] Ibid at 820; 323. It is the one point, within the *Logic*, at which phenomenological and logical dialectics touch.

[40] Ibid at 594; 29. And see above n.15.

[41] "Mathematical logic, which is nothing else but a precise and complete formulation of formal logic, has two quite different aspects. On the one hand, it is a section of mathematics treating of classes, relations, combinations of symbols etc instead of

and the notion of 'formal' embedded or inscribed in it. But it also emerged in and as a radical transformation of the tradition. What has been brought to the old formal logic is a method of formalisation which, in drawing on mathematical methods and reasoning, has encased 'the formal' of formal logic in a new, mathematical discipline.[42] Formal logic has been taken over by mathematics. I do not claim that Hegel's concept of 'formal' has won out, not the least because Hegel's thought contributed nothing to the transformation: it took place, as I said, somewhat to the side of that thought. But the transformation has seen both unanticipated results (antinomies), and new concepts (completeness, incompleteness, consistency, undecidability) which, applying to formal systems themselves, indicate a content of greater complexity than previously envisaged. Any formal mathematical logic 'in the spirit of Hegel' would, in order to be a formal mathematical logic, have to be constructed and presented in accordance with the requirements of that discipline and be open to the charge that it is not 'in the spirit of Hegel'. Leaving that debate to the proprietary minded, my point here is that the burgeoning complexity of formal mathematical logic is not inconsistent with Hegel's claim that

> this [his] notion of formal (*dieses Formelle*) must be regarded as posessing richer determinations and a richer content and as being infinitely more potent in its influence on the concrete than is usually supposed.[43]

The conceptual loss that I am asserting here deprives theory of a question that can and in my view should be put to understandings of Hegel as 'panlogicist', as regards its 'logicist' component. What conception of 'logic' and what concrete logical system for doing logic is being assumed?[44] Second, and as regards the 'pan', I think Butler is constrained by her arguments to pass over Hegel's assertion of the *independence* of the logical Notion from its modes of existence in nature and Spirit. Yet this is essential to the alternative conception of 'formal' that is in question. It

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numbers, functions, geometric figures etc. On the other hand, it is a science prior to all others, which contains the ideas and principles underlying all sciences" (K. Gödel, 'Russell's mathematical logic' (1944) in Kurt Gödel, *Collected Works*, vol.II, ed. S. Feferman *et al*, Oxford, Oxford University Press, 1990, 119–141 at 119).

[42] See Appendix, point 1, below.

[43] Hegel, above n.9 at 594; 29 [modified translation V.K.].

[44] Cf. Thomas Sören Hoffman, *Georg Wilhelm Friedrich Hegel: eine Propädeutik*, Wiesbaden, Marixverlag, 2004 at 381.

is an independence, *as constituted*, and *not* by the reductive abstraction involved in grasping what is essential to a matter in hand: not then by the concept of abstraction which Hegel offers in the place of the criticised Kantian notion, "the sublation and reduction of sensuous material to its essence".[45] There is no such material to be dealt with in logic. The constitutive abstraction is the objectification. It is thus, it seems to me, that the idea that Hegel wishes to comprehend everything about everything that exists — stones, states, situations, contingencies, discourses, emotions, attitudes, everything — in terms of the logical Idea is countered.

Still, there is something else to the charge of 'panlogicism' in this exchange: something that the term, taken less literally or analytically, marks. I think it to involve the charge of claiming to have deduced the One True System and I think that Hegel is in principle committed to this claim.[46] Further in taking Hegel's idea of a dialectical and speculative logical foundation of thought seriously I am associating myself with such a commitment. I also think it not inconsistent, indeed dependent on, an idea of 'logic' as an open dialectic of concept formation. In a world characterised by changeability, whatever domains of static truths may be established within it, a logic that would provide thought with the kind of foundation Hegel envisages cannot seal itself into a completed system. If

[45] "Abstract thinking, ... is not to be regarded (*zu betrachten*) as a mere setting aside of the sensuous material, the reality of which is not thereby impaired; rather is it the sublating and reduction (*das Aufheben und die Reduktion*) of that material as mere *appearance* to the *essential*, which is manifested only in the *Notion*" (Hegel, above n.9 at 588; 20–1). It might be objected that Hegel's selection of predication is reductive, but the place of natural language in Hegel's philosophy is not that of 'the material' to which he is referring here.

[46] The original charge of 'panlogicism', a term coined in the early twentieth century by Hermann Glockner, (as distinct from 'panlogism' which is an earlier and different charge) attributes to Hegel a form of idealism that is committed to a disembodied 'mind' as the substance or substratum or essence of being (*The Cambridge Dictionary of Philosophy*, Cambridge, Cambridge University Press, 1995 at 315). As charged by Laclau, Hegel's 'panlogicism' is his "project of presuppositionless philosophy" ('Constructing Universality' in *Contingency, Hegemony, Universality*, above n.2 at 306, n.2). From his fuller discussion (in the same volume, 'Identity and Hegemony' at 59f.) the "closed totality" of the absolute Idea beyond which "no further advance is possible", in combination with the necessary rather than contingent nature of Hegelian transitions, seems to constitute the gravamen of the charge. I think with Butler that the meaning of the term is very unclear. My formulation attempts to get at what actually is being objected to here.

it did, it would become redundant or lose its character as an organon for the production of objective insights and become a canon of how we ought to think: something like a catechism or diamat.

In principle; and in practice? First a further clarification of the principle; I am contemplating a conception of 'logic' as an open dialectic of concept formation: a dialectic within which the totalities ('universals') that concepts are and the totalisations over a finite or indeed infinite class of particulars (including concepts) that yield further 'higher order' concepts (such as the universal equivalent in Marx's general form of value[47] or transfinite numbers in Cantor's set theory[48]) extend, inexhaustibly, the logical realm. But now back to practice; does not Hegel close the logical realm, exhaust it in or in the name of the absolute Idea?

Yes or no: this staple of those for whom Hegel is panlogicist is like the repetition of Hegelian dialectic as a triadic movement of thesis, antithesis and synthesis.[49] One can put much scholarly exegesis against them, rustle up this reading and that reading, get into the endless, useless exchange of charge and complaint — misreading, misinterpretation, misunderstanding — that is the bubble-gum of academic discourse. They do not go away and I think they have less to do with Hegel than with questions of the commitments of those who are convinced that this is the truth about Hegel. I do not mean this dismissively. One can jump onto band wagons and perform well in current debates by following the fashions of philosophical and social theoretical discourses, but one cannot think freely and originally without commitments of a kind that make up the normativity of theory. Such commitments need not be dogmatic. They may be relativised to take account of competing commitments and to permit pragmatic moments within a theoretical praxis of justification and critique. Suspension of commitment for the purpose of following a theory or argument moving off from different commitments is a possible and valuable technique of theoretical engagement. A kind of translation between framings might

[47] Karl Marx, *Capital* I, trans. B. Fowkes, London, Penguin, 1976 at 157.

[48] An accessible account is given in Martin Davis, *The Universal Computer: the Road from Leibniz to Turing*, New York, W.W. Norton & Co., 2000 at 69–73.

[49] See e.g. I. Grattan-Guinness, *The Search for Mathematical Roots: Logic, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel*, Princeton, Princeton University Press, 2000 at 72; and in a quite different context Anne Bottomley, 'Shock to thought: an encounter (of a third kind) with legal feminism' *Feminist Legal Studies* 2004, 12: 1–37.

reveal that differences go more to expression than substance. But all this said commitments are part and parcel of serious theoretical work.

Hegel is committed to the independence of the logical realm from the realms of nature and social and cultural life (spirit).[50] It is a commitment to the possibility of constituting that realm by "complete abstraction": by an artifice of idealisation which I have little difficulty in crediting, albeit as an in-principle idea which would need mathematical techniques to be doable in practice. I have my own difficulties with the absolute Idea and they come back, as far as I can see, to the point that it is effectively a denial of that need. Hegel's conception of philosophy, as regards content and end, is one that classes it with art and religion as distinct from the particular sciences. But it has an elevated status within this class on account of its conceptuality.

> Philosophy has the same content and the same end as art and religion; but it is the highest mode of apprehending the absolute Idea, because its mode is the highest mode, the Notion. Hence it embraces those shapes of real and ideal finitude as well as of infinitude and holiness, and comprehends them and itself.[51]

I find it hard to say what actually the stakes are here. Value? Power? Authority? Degree of abstraction? All of these? More too: the genius of those individuals who can think thus abstractly and of those peoples whose cultural possession it is? But if this is Hegel's logocentrism and if it is inseparable from the ethnocentrism of the historical narrative that forms part of his thinking, I don't find logophobic solutions any advance in this predicament.

Nor, on my reading, does Butler. Rhetoric rather than logic is her tool, but just in re-staging the universal she is engaging the various conceptualisations of the universal in Hegel, staying short perhaps of the absolute Idea. That is not where our difference lies for holiness (*Heiligkeit*) as attributed to the absolute Idea is problematic for me too. The elevation of the divine above the mundane is a figure which Hegel reiterates from

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[50] Hegel, above n.9 at 586; 18: "[T]he Notion that is self-conscious and thinks pertains solely to spirit. But the logical form of the Notion is independent of its non-spiritual, and also its spiritual shapes. The necessary premonition on this point has already been given in the Introduction. It is a point that must not wait to be established within *logic* itself, but must be cleared up *before* that science is begun."

[51] Ibid at 824; 328.

beginning to end of his *Logic.* I think it compromises the emancipatory potential of his logical and metaphysical ideas and I think it implicated in that lordly contempt for slaves, tools and machines that thinks in terms of higher and lower forms of life and disparages the merely calculative. The task, as I see it, is to get away from this figure of elevation without, in a fit of pathos, dumping the *logos* into the dustbin of cultural prejudice and conceit. Again, however it is not such pathos in which I see Butler as indulging. My point as regards her, is rather that questioning the science of the *logos*, logic, must meet it where it is: in abstract, formal thinking, or rather, to keep hold of differing notions of 'formal' that I am seeking to elucidate, thinkings.

Philosophy, in the tradition inaugurated by Kant — that is, philosophy that holds to the idea of metaphysics as the urge to think the unconditioned (or absolute, or infinite, or totality) — adheres, not to all, but certainly to some of Hegel's ideas if it admits the experience of spirit in the development of mathematics and mathematical logic gained after Hegel's times.

5. Questioning authority

I am thinking of authority in general in terms of effective power of determination which lays claim to being justified. If it manages to bring that claim within its power, it gains a self-justifying, self-reproducing quality which strengthens it to a degree such that, at least as regards political authority, the need for a standpoint from which challenges to a constituted authority's claim to justification are made is widely acknowledged. What of the authority of 'logic'? I have reservations regarding the attribution of authority to logic. It seems to me that they strengthen a common and rhetorical use of claims to the logical necessity of an argument which neglect the relativity of any such necessity to a particular system of logic. I would prefer to say that logic neither has nor needs authority, being merely the consistent application (or construction) of a system of definitions, axioms and rules of inference which have been voluntarily adopted for some purpose or another. Often, I suspect, the appeal invokes, consciously or unconsciously, a claim to the universality of classical logic which I think is outmoded.

Still, the phrase is used by Georg von Wright in asking what gives to deontic logic the authority of Logic?[52] While I wonder whether, with his capitalised and I assume, totalised 'Logic', his question is quite straight faced, he explains that what is being asked after here is a *rationale* for specific principles of a deontic logic. In his view the question demands an answer. To give an answer, he asserts the necessity to step beyond "deontic logic itself" into discursive considerations of reasonableness relating norms to ends and a particular system of deontic logic to the Standard System.

The specific issue that von Wright is dealing with — controversy as to whether a reduction of deontic logic to alethic logic commits the 'naturalistic fallacy' — makes it clear that the question of the 'authority of Logic' is set into dispute in philosophy regarding the relation between theoretical and practical reason or, in the empiricist tradition, the 'is–ought' controversy. The ground then is that of Hegel's most fundamental disagreement with Kant. Already in the preliminary statement of his *logos* with its homage to and critique of Kant, Hegel's point of dissatisfaction is formulated. Kant's dialectical principle is too limited.

> [I]f no advance is made beyond the abstract negative principle of dialectic, the result is only the familiar one that reason is incapable of knowing the infinite; a strange result for — since the infinite is the Reasonable — it asserts that reason is incapable of knowing the Reasonable.[53]

Whether one speaks then of the 'authority of Logic' or, as does Hegel, of the 'authority of reason' may not make much difference. The ground takes in the politics of theory and, as it mainly concerns me, the politics of a critical (*qua* emancipatory) theory that is still struggling with the Greek heritage of post-Socratic philosophy and Aristotelian logic. On this ground, I am taking the standpoint of Hegel's 'absolute' beginning as regards 'the formal' of logic. That is to affirm the gap between logic/ontology and politics, to resist its dissolution (or 'healing') in ethical

[52] Georg von Wright 'Problems and Prospects of Deontic Logic' in *Modern Logic — a Survey*, ed. E. Agazzi, Dordrecht, D. Reidel, 1980, 399–423 at 408.

[53] Hegel, above n.16 at 56; 54. As von Wright elsewhere points out, one is looking here at different notions of practical inference. Though I would modify his formulation, he deftly grasps Hegel's thinking of practical inference as a (neglected) departure from both Aristotelian and Kantian paradigms ('On so-called practical inference' in *Practical Reasoning*, ed. J. Raz, Oxford, Oxford University Press, 1978, 46–62 at 47).

turns that take the paths of aesthetics or religion *and also* to resist the reduction of the problem to a hegemonic politics of discourse. The latter is my basic difference from Butler and Laclau and to it the constructive moment in formal logic is fundamental. It is that moment that establishes the logical Notion as independent of its modes of existence in nature and in social and cultural life and that brings practice — here practices of formal logical deduction and inference — into my notion of justification and thus 'foundation'.

Once a plurality of different systems of logic and of ways and frameworks of reasoning is acknowledged, the possibility of challenge to the justificatory claims of or associated with a particular system of logic is opened. It is a space that is especially fragile and vulnerable to the normative closures of pluralism and pragmatism. Let the standpoint from which the claims of a determining power to being justified are challenged be called that of 'the reasonable'. It could be named after freedom or truth or justice or right, but 'the reasonable' has the virtue of bringing things to a point. The idea I find in Hegel is that a self-justifying reason is also a self-critical reason and not quite in the same measure. It is weighted toward the latter as a reason that has learned the vanity of seeking to remove itself and its concepts from the play of reversals and surprises of the kind that Hegel traces in *The Phenomenology of Spirit*. These vain attempts to remove the being-at-odds-with-itself of reason from the reasonable are what Hegel calls 'formalism' and against which he proposes 'the formal' of his logic with its 'absolute' character. The plurality of notions of 'the universal' is now *within* Hegel's *Logic*. This is what Butler takes up but without the further sense of the 'absolute' of this formal logic having no 'outside', no 'beyond'. It is a logic of concept formation within a dynamic process: the ongoing activity of the thinking of finite, situated, embodied beings who seek to grasp that process in its excess, by thinking the infinite. My point, to echo Hegel, is that it is least of all, 'the formal', the formal notion of the reasonable, reason's self-conceptualisation, its self-objectification or extension, that should be excluded from a project of re-staging the universal by too limited a view of abstraction. The question here, to my mind, is practical-theoretical or technical. What is the method and means through which this self-justifying/self-critical reason is brought to recognition of its own points of impasse? This to my mind is a question of logic, of which logic, and of which approach to logic.

Let me recapitulate and then take further, the argument begun in the previous two sections. In following Hegel in his 'logical' turn of Kant's transcendental philosophy, but then looking to mathematical logic for the method and means of implementing that turn, it seems that conflicting notions of 'formal' are in play: the 'formal' that Hegel opposes, that which in the tradition of formal logic separates content and form, and the 'formal' which Hegel embraces with his idea of a science of logic. Because mathematical logic does continue the tradition of formal logic albeit shifting its discipline and practice from philosophy to mathematics, it is commonly thought that 'looking to' mathematical logic, so far as Hegel's logical ideas are concerned, is looking in the wrong direction. But this shift in discipline and practice is critical.

Hao Wang comments that "concern with forms, the formal and formalisation is central to the enterprise of mathematical logic". Discussing Gödel's incompleteness theorems, he goes on to say that they brought out the central importance of "the interplay of the formal and the intuitive even though the area is devoted to the study of the formal".[54] The point of this focus on the formal may just be, as Wang says, "to make precise the concept of formal and thereby be able to reason mathematically about formal systems". Mathematical and mathematical logical practice is pragmatic. Whichever system enables the pursuit of a particular endeavour will be used. The delicate and foundational question here is what lies between a practice which recognises the moments of indetermination in formal reason and ideological or normative standpoints of pragmatism and pluralism? I do not refuse the benefit of hindsight in approaching this question. For it is, and I use here a pragmatic criterion of outcome, the outcome of this way of thinking, namely that it has added "a new dimension to mathematics",[55] which makes me think that it is, despite Hegel's rejection of mathematical methods, not in principle alien to his conception of logic. It has something of the quality of a method "for the production of objective insights"; a method whereby potentialities immanent in thought can be brought out and articulated by thought making

[54] Hao Wang, *Reflections on Kurt Gödel*, Cambridge, Mass., The MIT Press, 1987 at 267.

[55] Hao Wang, *Popular Lectures on Mathematical Logic*, New York, Dover Publications, 1993 at 16.

itself its own object: by thought turning itself back on itself by means of itself.

I say 'something of' because Gödel's result as, so far as I can see, most of the work done in mathematical logic, is a result of and in classical logic. It seems to be on account of his technical virtuosity as a classical, formal, mathematical logician that Gödel managed to skirt the difficulties which classical logic has with handling self-reference and deliver his surprise. My point here is just that this row of adjectives — classical, formal, mathematical — suggests a quite particular logical system. What changes when 'classical' is dropped? I am saying that logic's disciplinary shift and the method of formalisation that accompanied it, brings a notion of 'formal' that is not wholly continuous with that which Hegel was opposing.

As a method, formalisation in logic and mathematics underwent a development which is inseparable from the development of mathematical logic and from developments in mathematics in the nineteenth century. The latter, in the later part of the century, saw a reconceptualisation of the very nature of mathematics itself. From being conceived as the science of quantity, the application of mathematical abstractions to the study of mathematical objects began to invest the abstractions with "a life of their own".[56] By 1847, George Boole was proposing a definition of mathematics as a general science of symbolic calculi (semiotics). Along a different path, Cantor in 1883 characterised mathematics as

> in its development entirely free ... and only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships ordered by definitions, to those concepts which have been previously introduced and are already at hand and established.[57]

The theme of abstractions acquiring a life of their own is also emphasised by Rózsa Péter (sometimes called 'the mother of recursive function theory'):

> Man created the natural number system for his own purposes .... But once created, he has no further power over it. The natural number series exists; it has acquired an independent existence.

[56] Jeremy Avigad and Erich H. Reck, '"Clarifying the nature of the infinite": the development of metamathematics and proof theory', Carnegie Mellon Technical Report CMU-PHIL-120, 2001, 1–53 at 8.

[57] Quoted from Avigad and Reck, ibid at 9.

> No more alterations can be made; it has its own laws and its own peculiar properties, properties such as man never even dreamed of when he created it. The sorcerer's apprentice stands in utter amazement before the spirits he has raised. The mathematician 'creates a new world out of nothing' and then this world gets hold of him with its mysterious, unexpected regularities.[58]

My point here is not just to support my suggestion of discontinuity with conceptions of the formal in the context of the old formal, classical logic. It is also to attempt to lay hold of and bring together two further thoughts. First, that Hegel's *logos* "the objectivity of illusion and the necessity of contradiction inhering in thought determinations" is, as a principle of intelligibility, one that certainly points to the demiurgos-like character of the agency of pure reason, but not one which, as Marx thought, posits the Idea as a demiurgos outside the realm constructed by thought thinking itself. I take Hegel, in his *Logic*, to be working with and within the human capacity for abstraction and idealisation in conceptual thought. The actual or completed infinite, as conceived and used by Cantor in creating his set theory, as distinct from the traditional and Aristotelian potential infinite, is illustrative of the capacity for abstraction and idealisation to which I refer.[59] That it first gave rise to a theory which became the fundamental discipline for the whole of mathematics; a "completely solid and sound" basis of mathematics as Poincaré at one time declared,[60] and then, as itself contradictory and in leading Russell to the discovery of his famous antinomy, caused "the whole building to rock",[61] is the (dramatised) phenomenon that leads into the second thought.

As I have said, Hegel's *Logic* is, as regards a thinking subject, subjectless. But the abstract *idea* of his *Logic* is a rethinking of the notion of objectivity emerging from *The Phenomenology.* This idea goes hand in hand with a thinking subject that knows itself as implicated in, not opposed to the 'objects' of its knowledge. It is the idea of a subject constituted in the

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[58] Rózsa Péter, *Playing with Infinity: Mathematical explorations and excursions* [1957], trans. Dr. Z.P. Dienes, New York, Dover Publications, 1976 at 22.

[59] The other example, used earlier, of a higher order concept, Marx's general form of value, is different in that it is a concept which comprehends an actual social development of practices of production and exchange.

[60] Abraham A. Fraenkel, Yehoshua Bar-Hillel, Azriel Levy, *Foundations of Set Theory* second rev. ed., Amsterdam, North-Holland Publishing Co., 1973 at 14.

[61] Péter, above n.58 at 229.

experience of a range of attitudes to its desires and their reversal. This thinking subject is, so to speak, paused or suspended, once the logical realm is entered. Its place is taken by method. But the experience of this subject in time is not 'cancelled' and that its experience, in the mathematical realm, is a further chapter in its phenomenology is my second thought. Thinking the infinite and its contradictions, and the desire for a system of complete and certain knowledge and its reversal, in the period from Cantor through to Gödel, play out as variations on Hegel's theme.[62]

As a layered narrative of the experiences of a journeying consciousness, *The Phenomenology* is a philosophical genre that resiles from an oppositional relation between narrative and conceptual discourse. The *Logic*, certainly, is inserted into the tradition of purely conceptual, that is *formal*, discourse, in which methods of abstraction take the place of narrative form.[63] That makes for a gap between the two works and it is, I think, for this reason that the reader is asked to think twice about the relation between them by Hegel's designation of *The Phenomenology* as the presupposition of a presuppositionless logic.

This thinking twice, this thinking between the phenomenological and the logical is the ongoing demand on thought which is my answer to the question posed above regarding pragmatism and pluralism. It should not be plastered over by words which offer verbal formulae in its place, whether these are Hegel's words ('speculative proposition') or words with current appeal ('aporetic', 'paradoxical'). Butler rests her discussion of Hegel on universality with just such a formula.[64] Gillian Rose, another inspiring reader of Hegel, does likewise with the repeated formulation of the absolute as that which must and cannot be thought.[65] It is, to my

[62] Like the failure of Frege's project, Gödel's incompleteness theorems have curious relations with tradition and change. As the use of mathematical methods to construct and investigate formal mathematical systems added a new dimension *to mathematics* and increased mathematical knowledge of formal systems, it radically undermined the idea of foundations and the means of securing foundations for classical mathematics that David Hilbert, at a certain end-point of Kantian ideas of foundation, proposed. See further, the Appendix to this paper below.

[63] Hegel, above n.9 at 588; 21.

[64] "The propositional sense of the copula must be replaced with the speculative one" (Butler, above n.2 at 24).

[65] Gillian Rose, *Hegel contra Sociology*, London, Athlone, 1981 at 42, 92, 204. Encapsulated in this repeated and developed assertion, is the view that Hegel's philosophy has no social import if the absolute cannot be thought, followed by the observation

mind, the failure to include that experience of spirit which came with the shift in the discipline and practice of logic from philosophy to mathematics and the accompanying loss or innocence of the tools of formal logic that constrains them.

So how does the relation in question look if it *is* thought twice about, now, almost two hundred years later? It is the relation between a theory of knowledge or justification and a logic. Whether and how Hegel would have taken up and thought that reversal of Leibniz' dream which showed up with the appearance of antinomies and incompleteness in (some) formal systems is and must remain a question. But to my mind, the recollecting totality that Hegel's journeying consciousness knows itself to be, can hardly be thought to have learnt or to remember anything at all if it thinks its ground will not again be pulled from under its feet.

That ground is Hegel's judgement regarding the means and method for attaining the 'complete abstraction' at which he aimed in his logic. The means is natural language. Philosophy, Hegel asserts, has the "right to select from the language of common life which is made for the world of pictorial thinking, such expressions as *seem to approximate* to the determinations of the Notion".[66] The method must be taken from philosophical not mathematical reasoning. Laclau finds, on the first point, a definitive argument against Hegel's logic being a logic at all.[67] It is not to my point to argue otherwise if by 'Hegel's logic' is meant the concrete product of Hegel's labours. It is Hegel's *idea* of and for a speculative and dialectical logic that will take the place of metaphysics that I am holding on to. I reject his judgement as overtaken as regards that very purpose. For, what emerges in mathematical logic, through formalisation, is a peculiar content, peculiar limits to the 'precisely defined' notion of the formal itself, which tear at the very way in which 'logic' is thought: tears at its own classical paradigm. This is a phenomenon that so far from refuting Hegel's idea renews the possibility of its realisation.

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that the absolute cannot be thought because the dichotomies of concept and intuition, theoretical and practical reason are not transcended. I owe a great deal to Rose's interpretation of Hegel, but I think that the juxtaposition of something that *must* be done (if Hegel's thought is to be relevant and useful) and its present impossibility too short. It sticks in the mystery of the speculative proposition which tries to say and unsay freedom's necessity.

[66] Hegel, above n.9 at 708; 177.

[67] Above n.2 at 63.

I am following through on Hegel's thinking to a point where, to my mind, there is a parting of the ways between 'philosophy' and 'logic' as the latter is now constituted and practiced. I do not see how philosophy can be denied the 'right' which Hegel takes it to have without losing its 'self'.[68] And who knows what Hegel's greater commitment was to: philosophy, his own *Dasein* as philosopher and the sanctity of his judgement regarding method and means, or his idea of a new, non-classical, dialectical and speculative, science of logic? If the latter, he might have blushed at a certain lack of courage of conviction in the power of dialectic to assert itself even in so barren a realm as that of numbers; a certain fear that Leibniz was right and that one needs the imprecision and ambiguity of a natural language to show thought at odds with itself.

I am following it through to a point where it is no longer a question of what Hegel thought or intended, but following it through on just that aspect of his idea of thought's logical foundation that is tied to the experience of spirit. The check, that which holds the absolute in Hegel's thinking of it away from the banality of absolutism, is that experience. Should one say then that the reasonable, reason's fetish form, drawn back into this process, is endlessly, infinitely revisable? In time I think so. But the thought here, and this is where 'fetish' is a misnomer, is the idea of a logic which as reasonable must also be actual: a presented system satisfying the demands of what is currently known of the discipline and practice of logic. Here Hegel failed, on his judgement and according to the demand he himself placed on the reasonable.

That is not to say, on the basis of a different judgement as to means and method, that his idea of a dialectical and speculative logic and the notion of thought's logical foundation that thinks between the phenomenological and the logical has failed or must fail or can only sustain itself in failing. It is to affirm the objectivity of illusion and the necessity of contradiction inhering in the determinations of thought as the principle of intelligibility of objectivity and necessity. It is also to argue for setting the formal science of this *logos* against the tendency of constituted and instituted power to bring the justification of its determinative power within the ambit of that power.

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[68] Cf. Jean-Luc Nancy, *Hegel: the restlessness of the negative*, trans. Jason Smith and Stephen Miller, Minneapolis, University of Minnesota Press, 2002, esp. 19–24.

In the Appendix, I attempt to communicate something of what, as an interdisciplinary researcher, I understand of the discipline and practice of formal mathematical logic. Here I make the plea that motivates this paper. It should be clear that what counts as logical necessity is not independent of the idea of logic involved and gains a formal but concrete sense in a system of logic. It is not only Hegel who rests neither with an abstract idea nor with the currently ruling paradigm. What logic is best suited to which scientific or technological endeavour is a question that is not strange to contemporary logical practice. May it not also be asked which logic is best suited to the endeavour of conceiving objective thinking from the premise that objects that are apprehended and comprehended by thought, whatever their genesis and corporeality, are formed by a subjective activity?[69] And further: is classical logic suited to reasoning which keeps account of its assumptions from the perspective that an assumption, once used in an act of inference might, like a dollar spent, be so to speak, used up and no longer available in the new situation thought finds itself in?[70] No doubt there are many reasons for the continuing hegemony of classical logic in both its philosophical and mathematical presentations and some may be 'good reasons' for what that is worth. It seems to me that since what counts as a 'good reason' must fall under the concept of the reasonable inhering in the way of reasoning formalised in a logic, it is worth very little without the kind of scrutiny given to the Idea of the good that differentiates Hegel's notion of practical inference from that of Aristotle and Kant, and from pragmatist and neo-pragmatist approaches to justification. Mathematics and mathematical logic has found a freedom in its forms of abstraction and idealisation that can mechanise and simulate itself in some degree. This is the present and there are no gods coming to save us from proprietary abuse of this power. Intervening into this situation, taking responsibility for it, resisting the impotence of beautiful souls and the indifference of 'hard nosed pragmatists' who are quite content with the current balance and conduct of power in the world, calls

[69] It should not be assumed that I am dismissing the pertinence of psychoanalysis and psychoanalytic theory to this question. On the contrary, I see that as also emerging through the experience of spirit following Hegel's times. My focus however is on the notion of objectivity. I would hope that the lacuna of focus coming from the demands of interdisciplinarity as I apprehend them might be met by collaborative work.

[70] See the Appendix to this paper, point 3, below, for the background of this question.

the humanities 'on the left' to take heed of conceptions and practices of formal logic.

As things now stand, so far as conceptions of logic are concerned, I have come upon this:

> [f]or us now, in the last quarter of the twentieth century, the nature of logic — that is, of the discipline of formal, deductive logic — is very largely unproblematic: it is a pure science devoted to the investigation and codification of relations of deductive consequence holding between sentences, or perhaps between the thoughts or propositions they express. And in this connection we understand the need to distinguish (proof theoretic) relations of syntactic consequence and (model theoretic) relations of semantic consequence; there is a general consensus as to how issues concerning, say, formal schemata, calculi, interpretation, truth, validity, consistency and completeness are related one to the other, and today we can be clearer than ever before about how, if at all, the subject matter of logic is related to that of the other disciplines like psychology, mathematics, set theory, ontology, epistemology or linguistics.[71]

Clarity, precision and mastery of the nature of logic to hand! Hurrah for 'us'! And this:

> In recent years we have witnessed a very strong and fruitful interaction between traditional logic on the one hand and computer science and Artificial intelligence on the other. As a result, there was urgent need for logic to evolve. New systems were developed to cater for the needs of applications. Old concepts were changed and modified and new concepts came into prominence. The community became divided. Many expressed themselves strongly, both for and against, the new ideas. Papers were rejected or accepted on ideological grounds as well as on technical substance.
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> In this atmosphere it seemed necessary to clarify the basic concepts underlying logic and computation, especially the very notion of a logical system.[72]

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[71] David Bell, *Husserl*, London and New York, Routledge, 1990 at 87.

[72] Dov M. Gabbay, Preface to *What is a Logical System*, Oxford, Clarendon, 1994 at v.

Disharmony and dissent — ideology no less! — among workers in the field. And this:

> [s]ince the time of Hilbert, no new foundational scheme has been proposed. Certainly people know too much to present a naive ontology of mathematics (and perhaps not yet enough to present a really challenging explanation of mathematical activity).[73]

And:

> [i]t has been a long time since philosophy stopped interacting with logic . . . .[74]

Which latter comment is what, were it possible, my plea would change.

APPENDIX

Non-expert observations of mathematical logic

The following makes something of the perspective of mathematical logic than I have gained as an interdisciplinary researcher available to the reader. It is bound to be a bit of a mess: a translation of methods and ideas from a discipline that has, from a certain necessity of abstraction, "gone beyond words", back into words! It is an absurd undertaking in its way. But I want to underline the sense in which this field, if technical, is also mundane: not mysterious, not a "more than human possession".[75] I also want to indicate a path taken into this field that is quite different from the standard introductions and undergraduate courses.

1. **Formalism**: In mathematical logic, 'formalism' is a term commonly applied to the product of a particular method of formalisation that developed as part of the emergence of the new discipline. In general, the term 'formalism' is synonymous with a range of other terms: calculus, formal system, formal theory, formal mathematics. In so far as formalisation is a general method, thus part and parcel of mathematical logic, this sense

[73] Jean-Yves Girard, *Proof Theory and Logical Complexity* v.I, Napoli, Bibliopolis, 1987 at 38.

[74] J.-Y. Girard, 'On the Meaning of Logical Rules I: Syntax Versus Semantics' in *Computational Logic*, ed. Ulrich Berger and Helmut Schwichtenberg, Berlin, Springer Verlag, 1999, 215–272 at 216.

[75] Hegel, above n.16 at 34: citing Aristotle on logic.

of 'formalism' is not negatively connoted. The contribution to methods of formalisation made within the Hilbert school in the first decades of the last century, and the proof theory or metamathematics it established — the logic of logic, as it has been called[76] — has been a focus of my research.

The method of formalisation developed in this school, proceeds, by means of a "completely symbolic language", through formal axiomatisation (which renders the primitive terms of the theory meaningless), to divest all other words used in deductions of their meaning, so as to yield an exhaustively defined 'object theory' (or 'formalism') for purposes of study. The formalisation and study of its product is done from within an informal metatheory.[77] Its point, so far as Hilbert's program was concerned, was to prove the consistency of the object theory (some part, eventually, Hilbert hoped, all of classical mathematics) by metamathematical means which, in being restricted to wholly uncontentious methods (so making no use of disputed rules such as *tertium non datur*) would be effective to secure the object theory (which might make use of such rules).

Thus proof theory or metamathematics was developed as means to a particular justificatory end, roughly, to place all of mathematics on a neo-Kantian (methodologistic) foundation that would secure its accomplishments (principally set theory) and 'the right' to use principles of classical logic in mathematics. The method in Gödel's hands, helped rather to undermine that aspiration. In one way of looking at it, it could be said that the master's tools were indeed used to dismantle the master's house. Without that procedure this proof would not have been possible and with this proof the last grand theory of foundations of mathematics, that is foundations of the kind that guarantee the truth of existing mathematical knowledge, collapsed. The demands on presentation of a formal logical system remain. Thus

> A system of symbolic logic must begin with a list of undefined symbols, a list of formal axioms, and a list of rules of inference.[78]

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[76] J.-Y. Girard, above n.73 at 10: but perhaps more helpfully "Proof theory = logic from a syntactic viewpoint" (ibid).

[77] Drawing on Stephen Cole Kleene, *Mathematical Logic*, New York, Dover Publications, 1967 at 198f.

[78] A. Church, 'The Richard Paradox' *American Mathematical Monthly* 41 (1934), 356–361 at 356.

That is to say the system is communicated by presentation of a "precise statement of the syntax of the formalism"[79] It will not be presuppositionless, but it may be the endeavour of setting up a formal logical system to both minimise assumptions and make them explicit as axioms, definitions or precisely formulated rules of the system. One could then say, that constructing a logical system has the character of constructing a game.

In foundations of mathematics, as distinct from the broader context of mathematical logic, 'formalism' has a different sense. It may be applied, approvingly, disapprovingly or neutrally to a theoretical standpoint within a debate precipitated by the emergence of antinomies and paradoxes at very basic levels of logic and set theory that took place in the early decades of the last century. In this context, 'formalism' and 'formalists (Hilbert school), 'logicism' and 'logicists' (Frege, Russell and Whitehead) and 'intuitionism' and 'intutionists' (Brouwer and Heyting) entered the lists of the kind of normative or ideological debate familiar in other theoretical endeavours.[80] The heat seems to have gone out of these debates from about the 1930's, although foundations of mathematics remains as an area of research. My impression is that the field expanded so considerably from the 1930's as to overtake these debates.

2. **Structural rules, logic*s* and logical practice.** Classical logic, while having been so radically transformed as to warrant the distinction between formal (philosophical) and symbolic (mathematical) logic,[81] maintains itself within the tradition of Aristotelian logic. It can be presented in terms of different axiom systems and rules and is expressible in various calculi, each of which is fully translatable into any other. In some such calculi (Gentzen sequential calculi) axioms are replaced by rules for the

[79] K. Gödel, 'Russell's Mathematical Logic" in *The Philosophy of Betrand Russell*, in ed. P.A. Schilpp (Library of Living Philosophers 5) La Salle, Illinois, Open Court Publishing, 1949 at 126.

[80] Although somewhat technical, and although historians of mathematical logic are now writing more finely tuned accounts of these debates, I think S.C. Kleene's classic text-book *Introduction to Metamathematics*, Groningen/Amsterdam, Wolters-Noordhoff Publishing/North-Holland Publishing Company, 1972, Ch.III a readable introduction.

[81] While Leibniz is an ancestor figure, the transformation began in the mid-nineteenth century with the work of Boole and the algebraists, gaining impetus and a somewhat different direction with Frege's *Begriffsschrift* (1879). The two streams merged in the 1930's. See Jean van Heijenoort 'Logic as Calculus and Logic as Language', above n.25.

introduction of logical constants ('operational rules') and rules of a different type, commonly termed 'structural rules'.[82] The types of rules are clearly distinguishable. The structural rules contain no logical constants ('and', 'or', 'if... then' etc.) and can be seen as governing the handling of assumptions only, in one of Gentzen's calculi.

Commonly, just four structural rules are listed. They embody properties of the deducibility relation in classical logic which are built into it prior to what may otherwise be regarded as its 'content', the logical constants. Non-classical logics may well include some of the structural rules, but a distinguishing feature of classical and non-classical logics is that the latter give up one or more of the structural rules whereas classical logic is a singular edifice. There is one and only one classical logic (although it is differently presentable and can be expressed in a variety of calculi) and it employs all of the structural rules. The field of mathematical or symbolic logic is thus divided between classical and non-classical (intuitionistic, many valued, quantum, dialetheic, dialectical, relevant, affine and others) logics. There are, certainly, polemical exchanges across that border. Apart from anything else, what is and is not 'logic' might be made an issue here.[83] But practice in the field sees logicians (and applied logicians such as theoretical computer scientists) employing whichever logic suits their purpose and regarding that as doing 'usual logic'. Here, as elsewhere, pluralism may be advocated as a favoured ideology. If so it looks to me as if the 'logic' debate is conducted on the familiar terrain of debate between 'conservatives' and 'liberals'.

3. **Contraction**: One of the structural rules that is of particular interest in the construction of a dialectical and speculative mathematical logic, has recently become the object of quite some activity in theoretical computer science. This rule (with the usual variability of terminology) is called

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[82] Gerhard Gentzen, 'Untersuchungen über das logische Schliessen', *Mathematische Zeitschrift* 41 (1934), 176–210 and 405–231. Translated by M.E. Szabo in *The Collected Papers of Gerhard Gentzen*, Amsterdam and London, North-Holland Publishing Company, 1969.

[83] Quine for example, with reference to Gödel's proof of the completeness of first-order predicate logic says: "This calculus [logic] is the basic department of modern formal logic; there are some who even equate it to logic, in a defensible narrow sense of the word". W.V. Quine, 'Kurt Gödel' in *Theories and Things*, Cambridge, Ma., Harvard University Press, 1981, 143–147 at 143. Terminology here is confusing since 'calculus' can be used to mean both a system of logic and its symbolic expression.

'contraction'. What it allows can be seen as dropping an assumption in a proof where that assumption has been used more than once. Standing behind this permission is an assumption of the re-usability of assumptions, or in other words, having an assumption once is as good as having it twice or more. Since 'resource consciousness' or 'good accounting' are desirable in constructing systems of logic suitable for computing (given practical limitations of time and memory) logical systems which drop contraction have been devised.[84]

As I understand contraction, in the reasoning that classical logic formalises, the possibility that assumptions may be used up in the 'act' of inference is excluded. Perhaps here something of the Hegelian spirit glimmers: a dynamic spirit for which, following an inference (transition), things are not just as before.

4. **Bivalence and its Restriction**: Perhaps this treatment of assumptions, in being consistent with ideas of unchangeable, eternal truths, supports these ideas and the ideal of truth abstracted from them. Perhaps it goes the other way, i.e. perhaps the rule is a formalisation of that ideal. Perhaps it goes both ways. I want to move on from here to the specific meta-logical assumption of classical logic itself, namely the assumption of bivalence, or truth-definiteness, or more fully of the validity of either–or reasoning *as applied to the truth values*, true and false. As Quine puts it:

> Bivalence is a basic trait of our classical theories of nature. It has us positing a true-false dichotomy across all the statements that we can express in our theoretical vocabulary, irrespective of our knowing how to decide them. In keeping with our theories of nature we have viewed all such sentences as having factual content, however remote from observation. In this way simplicity of theory has been served.[85]

Quine's purpose, in this article, is to point out the 'price' of bivalence. "We stalwarts of two-valued logic" he writes,

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[84] This is a technical area to which I know no introduction that is readable without specialist knowledge. The phrase "resource conscious" is taken from A.S. Troelstra, *Lectures on Linear Logic*, Stanford, Center for the Study of Language and Information, 1992 at 1.

[85] W.V. Quine, 'What Price Bivalence' *The Journal of Philosophy* LXXVII (1981), 90–95 at 94.

buy its sweet simplicity at no small price in respect of harboring of undecideables.[86]

The price, apparently, gives value for money: deductive power. Classical logic remains hegemonic. Its main rival, intuitionistic logic, though announced as a revolution,[87] turned out to be a rather moderate (and useful) reform.[88] But my point here is that this meta-logical assumption can be and is restricted in many non-classical logics. That is to say its universality is reigned in from all to some sentences. Or if logical bivalence is conceived as inherently universalist, then one would say it is given up, abandoned. It does not then follow that there are no sentences which are not either true or false in logics which restrict bivalence. Bivalence may be provable for *some* sentences by the logic in question. With such an abandonment, *accompanied by* an abandonment of the contraction rule, one gets a 'contraction-free' logic within which a certain form of contradiction (e.g. 'If A then not-A and if not-A then A') does not 'trivialise' the system (as does e.g. 'A and not-A'), that is, allow anything and everything to be proved by it. Again, the implementation of these acts of abandonment in a system of non-classical logic, takes shape in the presentation of the system. To William Rasch's playful question "To what is the law of excluded middle subject?"[89] my non-playful answer is: scrutiny and the possibility of being dropped!

5. **The abstraction axiom, antinomies and incompleteness**: Deductive power has been met. A formal logical system is also characterised by more or less expressive power. An abstraction axiom is basically a formal

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[86] Ibid at 91.

[87] By Hermann Weyl: see Davis, above n.48 at 96; see also Constance Reid, *Hilbert*, Berlin, Springer-Verlag, 1970, Ch. XVIII.

[88] The translation of a meta-logical assumption into theorems or rules of inference of a formal logical system is a tricky business. Intuitionistic logic 'gives up' excluded middle (or double negation which is equivalent to excluded middle within intuitionistic logic), and it may be said that, in a certain sense, contraction is restricted, because only one well formed formula is allowed in the succedent of a Gentzen sequent. But a well formed formula according to a schema 'not-(A and not-A)' (excluded contradiction) is deducible in intuitionistic logic. Some ingenious fiddling about with axiom systems and it turns out to be no less deductively powerful than classical logic. The moral here is that it does not give a lot of purchase on formal systems to talk about them in terms of the names traditionally given to principles of classical logic.

[89] William Rasch, *Sovereignty and its Discontents: On the Primacy of Conflict and the Structure of the Political*, London, Birkbeck Law Press, 2004 at 89.

axiom of concept formation and gives expressive power to systems including such an axiom. In natural languages with a subject–predicate structure, the construction of an object from a predicate, as in, for example, a move from 'is red' to 'redness' is illustrative of the operation permitted ('objectification'). It enables predication of an objectified predicate, such as 'redness'. That is, something further can be predicated of 'redness' whereas 'is red' cannot be used as a subject. Hence the usefulness of the rule. With an unrestricted abstraction axiom one has unlimited expressive power. Historically, its employment goes back to Frege; to his logicist project of reducing arithmetic to logic and his construction of the ideal calculus for that purpose, that is, (roughly! the technical details of Frege's system were otherwise given) classical predicate logic with abstraction axioms. The drawback is, as the failure of Frege's project showed, that in some cases, of which Russell's class is the classic example (the class of classes that are not members of themselves), when used in conjunction with classical logic, an abstraction axiom leads directly into full blown logical antinomy.

In classical logic, the abstraction rule is therefore severely restricted, although it is not given up.[90] But in a contraction-free logic, there is no passage from a contradiction appearing in the form 'If A then not-A and if not-A then A' to the form of contradiction 'A and not-A'. There is thus no need to restrict abstraction and such logics are said to allow unrestricted concept formation or unrestricted abstraction. When dealing with an object like Russell's class in such a logic, one will end up not with logical antinomy but with an 'undecidability' result.

An undecidability result: *the* undecidability result that passes so many lips, Gödel's incompleteness theorem, is derived in classical logic for consistent, formal theories (or axiom systems) containing some arithmetic. The first incompleteness theorem establishes that there is a sentence in the language of the theory in question, which is neither provable nor refutable in that theory. It is (roughly) a sentence that says of itself that it is unprovable. The second theorem shows that the consistency of the system cannot be proved within the system. What such results 'mean'

[90] The development of type theories as a means of avoiding such antinomies without entirely abandoning abstraction, is part of the history of mathematical logic. Type theories remain a feature of contemporary classical logic as one means of specifying restrictions on abstraction. There are also other, less restrictive, means.

philosophically is contested. It has been taken as an "excellent and beautiful example" of Hegelian dialectic[91] and it has been taken as a decisive refutation of Hegel.[92] I take it as an event of which the surprise is that a certain proposition is provable within the system for each and every natural number but is neither provable nor disprovable for the totality of all natural numbers. I fear that those who believe or even know that banning 'totalities' or 'closed totalities' is the way to deal with the threat of totalitarian political systems, will see my surprise as very naive. It is a risk I take.

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[91] J.N. Findlay, above p.36; Findlay's interpretation is backed by the truly remarkable accomplishment of a natural language version of the proof of the first theorem that appeared just twelve years after publication of Gödel's paper (J.N. Findlay, 'Goedelian sentences: a non-numerical approach' *Mind* LI (1942), 259–265).

[92] J.M. Bochenski, 'The general sense and character of modern logic' in Agazzi, above n.52, 3–14 at 14.

Philosophical Sanity, Mysteries of the Understanding, and Dialectical Logic

VALERIE KERRUISH AND UWE PETERSEN

ABSTRACT. Hegel's *Logic*, it is said, makes claims of a big order; claims which, as far as modern logic is concerned, cannot be upheld. Against this, the authors maintain that it is modern logic itself which has not come to grips with the very problems which lie at the bottom of Hegel's speculative philosophy and which show up in modern logic as paradoxes, incompleteness, and undecidability results. This paper is a plea for taking advantage of the failure of Frege's original conception of (higher order) logic for the development of a dialectical logic. It aims, in particular, at a younger generation of Hegel students, who are neither caught in the paradigm of logic as truth functional, nor reject wholesale deductive methods as inappropriate for the purpose of formulating a logic which aims at capturing its own factual content. The authors suggest that classical logic is to be given up in favour of a so-called 'substructural logic' which allows for unrestricted abstraction. Unrestricted abstraction, by way of its capacity to create all forms of self-reflexivity, is the source of an abundance of strange phenomena which lend themselves much better to Hegel's dialectic than to the dogmas of the understanding.

1. Hegel's Kantian Legacy

The point of Hegel's idea of a first philosophy that is at once a logic, metaphysics and ontology is to establish a logical foundation of thought-forms. In this he can be said to repeat Kant's question: how can subjective conditions of thought have objective validity.[1] Certainly, he sees Kant as having begun to turn metaphysics to logic. This turn however, and also its further development by Fichte, remained for Hegel seriously incomplete.

[1] Immanuel Kant, *Critique of Pure Reason*, trans. by Norman Kemp Smith (London: Macmillan, 1933), pp. 120–124.

> The critical philosophy did already turn metaphysics into logic, but for fear of the object it, like the later idealism … gave the logical determinations an essentially subjective meaning (*Bedeutung*); thereby they remained at the same time afflicted with the object they had fled, and a thing-in-itself, an infinite impetus, remained with them as a beyond (*SW* v.4, p. 47; *SL*, p. 51).[2]

Hegel's idea then was to complete this turn by reconceiving Kant's thing-in-itself as an abstraction or extension of reason: as the Reasonable.

Unlike Kant, who in his *Prolegomena to any Future Metaphysics* declares that only Hume's doubts "can be of the smallest use"' in the "perfectly new" science of metaphysics he had established,[3] Hegel's idea of metaphysics is not confined to the epistemological concerns of modern philosophy but also draws in the concerns of ancient and medieval metaphysics with content and substance.

> The objective logic … takes the place of the former *metaphysics* … — If we show consideration for the last form of the development of this science, then firstly it is immediately the *ontology*, the place of which is taken by the objective logic, — the part of that metaphysics which was meant to investigate the nature of the *ens* in general … But then the objective logic also comprises the remaining metaphysics in so far as this attempted to grasp with the pure forms of thought particular substrata, initially taken from figurate conception (*Vorstellung*), the soul, the world, God … (*SW* v.4, pp. 64–65; *SL*, p. 63).

Logically dealt with, according to Hegel, these forms of thought are freed from their submergence in self-conscious intuition and its substrata in 'figurate conception' (*Vorstellung*), that is, conception that is reliant on the myths and metaphors of sensuous perception.[4] Pre-Kantian metaphysics

[2] All quotations from Hegel are our translations, based on the fourth edition of the *Jubiläumsausgabe of Hegels Sämtliche Werke* (Stuttgart-Bad Cannstatt: Friedrich Frommann Verlag, 1961–8) cited hereafter as '*SW*' v. followed by the initials of the English translations noted. Here: *Science of Logic*, trans. by A. V. Miller (Atlantic Highlands, NJ: Humanities Press International, 1969). Cited hereafter as '*SL*'.

[3] Immanuel Kant, *Prolegomena To Any Future Metaphysics That Will be Able to Come Forward As Science*, trans. by Paul Carus, rev. by James W. Ellington (Indianapolis: Hackett Publishing Company, 1977), p. 7.

[4] "The myth is always a presentation which uses sensuous mode, introduces sensuous pictures, which are suited for the presentation, not to the thought; it is an

(though not, in their speculative moments Plato and Aristotle) omitted to do this and consequently

> incurred the just reproach of having employed these forms *without critique*, without a preliminary investigation, if and how they were capable of being determinations of the thing-in-itself to use the Kantian expression or rather of the Reasonable. The objective logic is therefore the true critique of them a critique which ... considers them themselves in their specific content (*SW* v.4, p. 65; *SL*, p. 64).

Reconstruction of the object, within a philosophical context in which historically it has been placed over against the subject entails reconstruction of the idea of subjectivity. Hegel's logic contains a third part, the doctrine of the Notion which he terms 'subjective' in the sense of being concerned with the subject itself: not the human subject but the living being of reason. In terms then of ideas of the subject and the subjective as opposed to the object and the objective, Hegel's logic is subject-less. It is not a phenomenology of spirit or consciousness. It is, or rather claims to be, a demonstration of how, taking nothing from outside, the totality of all determinations of pure thought, is derivable. In this, the science of logic takes a circular path which leads back to Being. This starting point however, is now enriched. It has been discovered that it contains all that succeeds it within itself. It has been 'ensouled by the method' and *acts* now to constitute the beginning of and for a new science.

> By dint of the demonstrated nature of the method, the science presents itself as a *circle* coiled in itself, into the beginning of which, the simple ground, the mediation coils back the end; in the process this circle is a *circle of circles*; for every single link, as ensouled by the method is the reflection into itself, which, in returning into the beginning is at the same time the beginning of a

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impotence of the thought, which does not yet know for itself how to hold on to itself, get by with itself. The mythical presentation, as older, is presentation where the thought is not yet free: it is pollution of the thought by sensuous form; this cannot express what the thought wants. It is appeal, a way of attracting, to occupy oneself with the content. It is something pedagogical. The myth belongs to the pedagogy of the human race. When the notion has grown up, it is no longer in need of it." *SW* v.18, pp. 188–9; G. W. F. Hegel, *Lectures on the History of Philosophy* (hereafter cited as '*LHP*') v.2, trans. by E. S. Haldane and Frances H. Simon (Lincoln: University of Nebraska Press, 1995), p. 20.

> new link. Fragments in this chain are the individual sciences, each of which has a *Before* and an *After*, or, more strictly speaking, only *has* the Before and in its very own ending *shows* its *After* (*SW* v.5, p. 351; *SL*, p. 842).

This 'simple ground' is not, as in Kant, the transcendental unity of apperception or, as in Fichte the ego or 'I'. It is logical *abstraction* in the sense of passing from a predicate to its extension — a sense of abstraction commonly termed 'reification'. Outrageously, Hegel's logical project claims to find within and by means of speculative reason, a logic that is not just a canon of judgements but an '*organon* for the production of *objective* insights' (*SW* v.5, 23; *SL*, 590). That is to say Hegel's logic claims not only to be *truth preserving* but to be *truth generative.* It involves nothing less than an attempt to derive from within thought, not only the *validity* and *value* of its categories for 'objective truth' (pace Kant) but also their *substance* and *content.*

In the face of the modern transformation of logic into an essentially mathematical discipline, is it open to read Hegel's logic *as a logic*? If Hegel's logic cannot be read as a logic then either a methodologistic (neo-Kantian) or broadly hermeneutic interpretation must be the best interpretive bet. Yet thinking about thinking, is evidently self-referential. Some will say that we 'ought not' engage in so silly an enterprise. Such is Carnap's doctrine of the "confusion of spheres",[5] strongly modelled after Russell's simple theory of types. But there is still the possibility that the contradictions and paradoxes of self-reference have epistemological significance and it is this possibility that we want to open and leave open.

If Hegel's idea of his own philosophy as grounded in a speculative logic is dismissed, then he will be interpreted within a framework of thought for which classical logic continues to play its particular truth preserving role. This role is premised on truth definiteness, that is the validity of either-or reasoning as applied to the truth values true and false.[6] Far from regarding this form of reasoning as canonical, Hegel links it to the

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[5] Rudolf Carnap, *The Logical Structure of the World. Pseudoproblems in Philosophy* (London: Routledge and Kegan Paul, 1967), pp. 53–54.

[6] In other words, the assumption that every closed sentence takes exactly one of the truth values *true* or *false.* Cf. n. 34 on p. 37 below regarding Pinkard's 'Reply to Duquette'.

Dogmatism of pre-Kantian metaphysics and claims to be dedicated to breaking up its stases.

> The struggle of reason consists in that, to overcome that which the understanding has fixated (*SW* v.8, § 32Z; *Enc*).[7]

This does not necessarily entail a denial of the validity of such reasoning, at least within a limited sphere. Hegel makes a distinction, within thought, between speculative reason and the understanding and he both assigns either-or reasoning to the latter and subordinates it to reason.

> *The theorem of the excluded middle* is the theorem of the determinate understanding which wants to keep the contradiction away from itself, and in so doing commits the very same (*SW* v.8, § 119Z; *Enc*).

The understanding is an essential moment of thought but, and not the least because of formal constraints of its valid exercise, Hegel does not regard it as adequate to philosophical cognition of truth. The mystery for the understanding is its own role.

The distinction between reason and the understanding is taken over from Kant, who had refined and extended use of the term intellect (*Verstand*) in the Wolffian school to apply to the general faculty of cognition as distinguished from *ratio* (*Vernunft*), or the power of seeing the connection between things. Reason, for Kant, as a faculty of *principles*, itself creates concepts, or more strictly Ideas, that are transcendent, that is are not taken from the senses via intuition (*Anschauung*) or from the understanding which gives conceptual unity to intuition through the application of its pure forms, the categories.[8] Hegel's notion of reason, in its departure from Kant, is fundamental to the issue between them. He does not depart from Kant's idea that reason, as the faculty of the unconditioned, that is, thought in its metaphysical exercise, seeks the totality, the unconditioned, the Idea.

> It was only by Kant that the distinction between Understanding and Reason has been pointed out decisively and determined in *such way* that the former has as its object the finite and conditioned, and the latter the infinite and unconditioned. Although it

[7] *Hegel's Logic: Part One of the Encyclopedia of the Philosophical Sciences*, trans. by William Wallace (Oxford: Clarendon Press, 1975). Cited hereafter as '*Enc*'.

[8] Wallace, 'Notes and Illustrations', *Enc.*, p. 310.

> is to be recognized as a very important result of the Kantian philosophy that it has asserted the finitude of the merely empirically based knowledge of the understanding and described its content as *appearance*, it is still not to be stopped at this negative result and the unconditionedness of the reason is not to be reduced to the merely abstract, the difference excluding, identity with itself. ... The same then also applies to the *Idea*, on which Kant did bring back honour insofar as he vindicated it in contrast to the abstract determinations of understanding or even merely sensuous representations (the like of which one may well be in the habit of calling ideas in ordinary life) of the reason, but with regard to which he likewise stopped with the negative and the mere *ought* (*SW* v.8, § 45Z; *Enc*).

Where Hegel *does* depart from Kant concerns the principles for reason's exercise. These principles are the concern of Kant's transcendental logic and of Hegel's dialectical or speculative logic.

2. Hegel Interpretation and Logical Illiteracy

It is one thing for an Hegelian or neo-Hegelian scholar faced, as a philosopher or social theorist without actual competence in modern logic,[9] with the still powerful and still dominant paradigm of classical logic, to take the path of prudence and read Hegel's logic as a form of hermeneutics or as a logic in the broader sense of a method and manner of reasoning.[10] Here at least Hegel's distinction between understanding and reason can be preserved. It is another thing to say that modern logic has ruled out the very value and sense of this distinction as Hegel conceived it, and with that any 'sane' acknowledgement of the critique of understanding or reflective reason from which Hegel's idea of speculative reason proceeds. This second alternative is proposed by Allen Wood. Hegel's ethical theory,

[9] By which we mean the ability to carry out proofs in logic, not just to cite results.

[10] This position is taken, for instance, in Terry Pinkard, 'A Reply to David Duquette', in *Essays on Hegel's Logic*, ed. by George di Giovanni (New York: New York University Press, 1990), pp. 19–20 (cited hereafter as 'Reply to Duquette'): "First, I want to argue that Hegelian dialectic does not challenge ordinary logic. Second, I want to suggest at least that Hegel's Logic should not to be taken strictly as a logic at all but only as an understanding of philosophical explanation."

in his view, has great merit if only it is taken "as philosophical sanity" judges most promising: in "the understanding's way".

> Viewed from a late twentieth-century perspective, it is evident that Hegel totally failed in his attempt to canonize speculative logic as the only proper form of philosophical thinking. ... When the theory of logic actually was revolutionized in the late nineteenth and early twentieth centuries, the new theory was built upon precisely those features of traditional logic that Hegel thought most dispensable. In light of it, philosophical sanity now usually judges that the most promising way to deal with the paradoxes that plague philosophy is the understanding's way. Hegel's system of dialectical logic has never won any acceptance outside an isolated and dwindling tradition of incorrigible enthusiasts.[11]

It is certainly hermeneutically odd for an interpretation of Hegel's ethical thought to be made within a way of thinking that excludes the understanding of philosophical thought which he was attempting to communicate. The 'understanding's way' is an ambiguous phrase. We would not however, and for reasons which will shortly become apparent, press any norm of hermeneutic method here to the point of proscribing such endeavours. It is not just a curiosity that analytic philosophers keep on producing 'sympathetic' interpretations of writers and texts whose most basic ideas they abhor. If Hegel's idea of the foundations of his philosophical system are dismissed, then so also is his idea of reason and the critique of the understanding on which it rests. This is just to Wood's point. It is part of a politics of the colonisation of metaphysical sense by common sense: a politics that *authorises* itself by allusion to modern logic.

Modern logic, quite simply, does not give this authority. It cannot decide the *philosophical* question that is in issue, namely how logic, metaphysics and ontology are related. Insofar as metaphysics is an inquiry into truth and human capacity for knowledge it includes epistemology. Whether or not it should confine itself to epistemology, leaving ontology more or less aside and deferring questions of ideals or values to a separated exercise of practical reason concerned with moral, legal and political philosophy, is one particularisation of that question. How it is answered

[11] Allen W. Wood, *Hegel's Ethical Thought* (Cambridge: Cambridge University Press, 1990), p. 4. Cited hereafter as 'Wood'.

turns on a set of questions which Wood forecloses. *Do* the logical paradoxes 'plague' philosophy, that is, do they threaten the healthy exercise of reason, or do they threaten the self-satisfaction of the understanding in its blindness to its own role? Might they not be *constitutive* of philosophical thought? One does not have to be Hegelian to acknowledge the latter possibility. It is part of the history of western philosophical thought, a point that has not gone unnoticed by logicians.[12] And since we now draw logicians into the philosophical question, it might be reasonable if not philosophically 'sane', to re-open the question of what 'the most promising way' to deal with the logical paradoxes *is*, from a logically competent perspective.

From such a perspective, Wood's statement needs a triple supplement. In the first place, it is pretentious to talk of precision when it comes to the "features of traditional logic" and that quite independent of Hegel's dealing with them. Even in modern philosophy of logic the issue of what a principle of logic is, is not always sufficiently clear. In particular discussions about non-classical logics are prone to suffer from a lack of awareness in this respect.[13] Apart from that, modern logic extends and revises the

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[12] Thus Fraenkel *et al*, set theoreticians, comment on Russell's antinomy. "To be sure, Russell's antinomy was not the first one to appear in a basic philosophical discipline. From Zenon of Elea up to Kant and the dialectic philosophy of the 19th century, epistemological contradictions awakened quite a few thinkers from their dogmatic slumber and induced them to refine their theories in order to meet these threats. But never before had an antinomy arisen at such an elementary level, involving so strongly the most fundamental notions of the two most 'exact' sciences, logic and mathematics." A. A. Fraenkel, Y. Bar-Hillel and A. Levy, *Foundations of Set Theory* (Amsterdam: North-Holland Publishing Company, 1973), p. 2.

[13] 'The principle of bivalence', for instance, is being given up in many ways, usually however without ever asking whether there might be something else that takes its place. The issue here is similar to that of the postulate of the parallels: it may come in a guise that is not readily recognisable as, for instance, a claim about the sum of the angles in a triangle. In some systems of logic, the 'principle of bivalence' can take the guise of any of the following formulas: $A \vee \neg A, \neg(A \wedge \neg A), (\neg A \rightarrow A) \rightarrow A$, $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B), \neg\neg A \rightarrow A$. In fact, replacing I. 3) in the list of formulas in D. Hilbert and P. Bernays, *Grundlagen der Mathematik* I (Berlin, Heidelberg, New York: Springer-Verlag, 1968, zweite Auflage), p. 65, by any of the first three of the foregoing wffs provides an axiomatisation of classical propositional calculus. The fifth and last of the foregoing wffs is characteristic of intuitionistic logic and shows little similarity to what is commonly called *tertium non datur*.
P.S. Since this was first written, we learned from one good referee of the present paper, who was said to have some logical expertise, that intuitionistic logic "does not have

common logic of Hegel's day so extensively that there is no precise mapping between the two.[14] The philosophical question of whether antinomies are to be dealt with in 'the understanding's way', is historically an issue between Kant and Hegel, both of whom were working with the old common logic. That this had hardly altered since Aristotle is a point which both mention. Kant takes it as proof of its soundness. Hegel considers it ripe for revision. Analogies, certainly can be made, but analogies are not precise.

Second, there is a considerable difference between 'dispensing' with features of traditional logic and *restricting* them to thought at the level of the understanding. Taken together these two points persuade some Hegel scholars that it is a myth to say that Hegel denied the 'law of non-contradiction'.[15] We are less concerned with debates concerning Hegel interpretation than with specifying two questions. First, what is involved in, and what kinds of logic come from restricting features of classical logic? Second what is required to read Hegel's *Logic as a logic*? We deal with both questions below, adding a little context from the history of logic, but a preliminary response to the second question, may be helpful here. To read Hegel's *Logic* as a logic, does *not* require commitment to the view that Hegel presented a logical derivation of the categories which has been or could be translated into the formal language of a modern logic. It is rather to see how the occurrence and significance of contradictions in thought that has itself and its own determinations as its objects, lies at the core of Hegel's extension and radicalisation of Kant's transcendental logic.

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excluded middle nor the other formulae listed" here. We have to admit that, in writing this note we did not sufficiently anticipate the possibility of such a response. In face of it our breath is clearly wasted.

[14] *Cum grano salis,* traditional (Aristotelian) logic may be regarded as monadic predicate logic. Cf. J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* (Oxford: Clarendon Press, 1957); also D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, fifth edition (Berlin, Heidelberg, New York: Springer-Verlag, 1967), pp. 57–63); and A. Tarski, *Einführung in die mathematische Logik*, (Berlin: Julius Springer, 1938), p. 46.

[15] Robert Hanna, 'From an Ontological Point of View: Hegel's Critique of the Common Logic' in *The Hegel Myths and Legends*, ed. by Jon Stewart (Evanston, Illinois: Northwestern University Press, 1996), pp. 253–281; Robert Pippin, 'Hegel's Metaphysics and the Problem of Contradiction', *op. cit.*, pp. 239–252.

The third point is more substantial and goes to the analogies between the traditional form of logic and modern mathematical logic that can justifiably be made. It is certainly the case that the 'revolution' in logic that took place in the late nineteenth and early twentieth century, owed nothing at all to Hegel. When the theory of logic was revolutionised, it was certainly not in the spirit of Hegel. It did not question classical logic and, in the work of Gottlob Frege, which made explicit the crucial shift from a concept to its extension, it ran straight into antinomies. These antinomies have come to be known as the *logical paradoxes*. Essentially, they are the kind of paradoxes, such as the Liar, that have been classified as shallow sophistries since the times of Aristotle. Moreover, 'solutions' to the modern paradoxes supplied by modern logic are often considered as artificial and unilluminating, at least by those who favour a different one. This does not vindicate Hegel, but it calls for more caution towards the kind of late twentieth century viewpoint evoked by Wood.

The question, to which Wood never advances, of what the significance of the antinomies into which Frege ran is for logic *and* philosophy, arose *within* logic, within a few decades of the 'revolution' to which Wood refers.

> Logical coercion is most strongly manifested *in a priori sciences*. Here the contest was to the strongest. In 1910 I published a book on the principle of contradiction in Aristotle's work, in which I strove to demonstrate that that principle is not so self evident as it is believed to be. Even then I strove to construct non-Aristotelian logic, but in vain.
>
> Now I believe to have succeeded in this. My path was indicated to me by the *antinomies* which prove that there is a gap in Aristotle's logic. Filling that gap led me to a transformation of the traditional principles of logic.
>
> ... I have proved that in addition to true and false propositions there are *possible* propositions, to which objective possibility corresponds as a third in addition to being and non-being.[16]

Such was Łukasiewicz' position in 1920. As regards logic, the question was still alive and unsettled more than forty years later:

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[16] Łukasiewicz, *Selected Works*, ed. by L. Borkowski (Amsterdam: North-Holland Publishing Company; and Warszawa: PWN-Polish Scientific Publishers, 1970), p. 86.

> the set theoretical paradoxes ... are a very serious problem, not for mathematics, however, but rather for logic and epistemology.[17]

And we find Myhill reaffirming another twenty years later,

> Gödel said to me more than once, "There never were any set-theoretic paradoxes, but the *property-theoretic* paradoxes are still unresolved",[18]

adding that

> the Fregean concept of property is inconsistent *with classical logic*. So if we want to take Frege's principle seriously, we must begin to look at some kind of nonclassical logic.[19]

We leave the assessment of the set theoretical situation to set theorists. Our point is that these comments, from within logic, hardly authorise a cavalier dismissal of antinomies. On the contrary, philosophy might, on pain of signing its own certificate of irrelevance, need further and better details regarding them.

Frege's project was an attempt to reduce arithmetic to higher order logic, that is, in modern terms, the ideal calculus, or classical logic with abstraction axioms. Such axioms, roughly, allow the formation of a concept, (e.g. redness) from a predicate (e.g. is red). While Frege expressed some doubt about whether they satisfied the requirements of purely logical axioms, he confessed himself unable to conceive numbers as objects without them. The problem was (and is) that within classical logic, these axioms lead into antinomies. When this was pointed out to Frege by Russell, Frege regarded his life work as having failed.

> Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished.
>
> This was the position I was placed in by a letter of Mr Bertrand Russell, just when the printing of this volume was nearing its completion. It is a matter of my Axiom (V). I have never

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[17] K. Gödel, 'What is Cantor's Continuum Problem?' in *Philosophy of Mathematics*, ed. by P. Benacerraf and H. Putnam (Englewood Cliffs: Prentice-Hall, 1964), pp. 258–273 at p. 262.

[18] J. Myhill, 'Paradoxes', *Synthese* 60 (1984), pp. 129–143 at p. 129.

[19] *Ibid*, p. 130.

> disguised from myself its lack of the self-evidence that belongs to the other axioms and that must properly be demanded by a logical law. And so in fact I indicated this weak point in the Preface to Vol i (p. VII). I should gladly have dispensed with this foundation if I had known of any substitute for it. And even now I do not see how arithmetic can be scientifically established; how numbers can be apprehended as logical objects, and brought under review; unless we are permitted — at least conditionally — to pass from a concept to its extension.[20]

For Frege this was the failure of a project. But what, precisely, did fail, is not clear. In the 'Nachwort' to his *Grundgesetze der Arithmetik* we find Frege pondering over whether the law of excluded middle would have to be restricted, or whether there are cases where we are not entitled to speak of the extension of a concept. Is it asking for too much to see an analogy here between Hegel's and Kant's ways with the Antinomy of Pure Reason?

For modern logic Frege's accomplishment was substantial. It was the accomplishment of what has been called the ideal calculus[21] and this, as noted is *classical* logic, most notably relations and quantifiers, with abstraction axioms.[22] What links Hegel to Frege, or to put that another way, which non-classical logics might be termed dialectical in Hegel's sense of that term, will obviously depend on how Hegel's idea of dialectic is interpreted. In the interpretation presented here, it is abstraction, passing from a concept or predicate to its extension for the purpose of constructing an object of reason: metaphysical reason in Hegel's endeavours, arithmetic in Frege's. As Frege perceived, the failure of his system turned on his *Grundgesetz* V which was introduced to govern the equality of *Werthverläufe*, that is, the extensions of concepts. In the long run, this is what abstraction, as a logical operation, comes down to. Abstraction, in some

[20] P. Geach and M. Black, *Translations from the Philosophical Writings of Gottlob Frege* (Oxford: Basil Blackwell, 1970), p. 234; G. Frege, *Grundgesetze der Arithmetik* v. 2 (Jena: H. Pohle, 1903), p. 253.

[21] For instance, by Fraenkel and others, *Foundations of Set Theory*, above n. 12, p. 154.

[22] More generally, it involved the analysis of the logical components of mathematical reasoning.

cases leads into contradictions and it was just such a contradiction, a 'vicious self-reference' that Russell pointed out to Frege.[23] This, in classical logic and via the rule of excluded middle ($A \vee \neg A$, or some equivalent formulation, cf. n. 13 on p. 32) is a total failure of truth preservation because the contradiction allows anything and everything to be proved.

Somewhat ironically, this situation confronts us with an *either—or* alternative: *either* to preserve classical logic and restrict abstraction (for example, through type distinctions such as Russell proposed in order to find a way around his antinomy) *or* to abandon classical logic and restrict the assumption of truth-definiteness that makes contradiction so unpalatable (in allowing anything and everything to be proved). The first way is probably what Wood means by 'the understanding's way'. It stays within classical logic and restricts abstraction. Of course, "philosophical sanity now usually judges that the most promising way with the paradoxes that plague" Frege's (higher order) logic is to sacrifice the general assumption of the existence of an extension to each and every concept if it has occurred to it that there is a problem. Mathematical logical enterprise is less confined. There are several non-classical logics. All dispense with truth-definiteness, where that is understood as a meta-logical assumption of the validity of either-or reasoning, *as applied to the truth values*, true and false. This is simply what makes them non-classical logics. But they dispense with truth-definiteness in different ways. They might introduce third or further values (as in Łukasiewicz' logics), they might allow cases in which a sentence is both true and false (as in paraconsistent logics, 'dialetheism') or they might aim at allowing unrestricted abstraction by directly dispensing with a 'logical law' in a particular axiomatisation of logic. Such logics may be said to have 'dispensed with' the meta-logical assumption of truth definiteness and might, in light of the link made above, be termed dialectical in an Hegelian sense.[24]

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[23] Strictly speaking, following Frege, it is not possible to predicate a predicate; but via abstraction a predicate can be objectified, and this objectification can then be predicated. For example it makes no sense of any kind to say 'Is red is red' but it can be said 'Redness is red'. In this case, the self-reference brought about by abstraction ('red' predicated of 'redness'), causes no problems. Russell's antinomy concerned the set of all sets which do not contain themselves as elements.

[24] Classical logic *can* be, indeed has been restricted. The possibility of restricting classical logic in such a way as to have unrestricted abstraction available (that is to include Frege's *Grundgesetz* V axiom or its equivalent) is simply not contentious, at

Or they might not. There are all manner of issues and, for that matter, non-issues here. Which non-classical logics *are* dialectical in an Hegelian sense?[25] Are non-classical logics a threat or a complement to classical logic?[26] And then, what does any of this matter? The irony, pointed out above, is that logic cannot take us further with the philosophical questions in issue here: the significance of contradictions in thought that has itself and its own determinations as its objects. The very formality of the either-or of the methods of avoiding Russell's antinomy leaves this question untouched. In that sense we reach a limit of logic's authority. To go further here, in natural language, we must go back to Hegel's issue with Kant — his extension and radicalisation of Kant's transcendental logic — as the classical discussion in modern philosophy on the significance of antinomies in the *a priori sciences*, with what has just been canvassed in mind. That is to say, the problem of antinomies in modern philosophy, while historically an issue between Kant and Hegel, is not *just* an issue between Kant and Hegel and it is not *just* an amusing pastime for speculative philosophers. What Wood does not mention is that 'shallow sophistries', such as the Liar, still plague higher order logic.

This brings us to the point of reading Hegel's *Logic* as a logic. We might recall, to begin with, what Hegel said in his *History of Philosophy* regarding Eubulides' sophisms:

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least amongst logicians. It was first established about 1950 independently by Fitch and Ackermann; cf. K. Schütte, *Beweistheorie* (Berlin, Göttingen, Heidelberg: Springer-Verlag, 1960), p. 333 for historical notes, and chapter VIII (pp. 224 ff) for technicalities.

[25] Within a philosophically realist framework, contradictions are located in reality and a non-classical logic that results is dialectical in the sense of dialetheic, that is, it allows that in certain cases, A and ¬A may both be true. See e.g. G. Priest, *Beyond the limits of thought* (Cambridge: Cambridge University Press, 1995), pp. 3 f. This is a philosophy of the limit and is opposed to the more scandalous view that contradictions are not just brute metaphysical facts to which a logic must conform, but are constitutive of the determinations of pure thought.

[26] Some modern logicians, who may be seen as having contributed to the development of a non-classical logic (Kleene and Kripke, for example), remain committed to an idea of truth consistent with classical logic. While working with three 'truth values', the third value ('undefined') is not an *extra* truth value. It is not on the same level as true and false and is not introduced on the assumption that classical logic does not generally hold. Cf. S. C. Kleene, *Introduction to Metamathematics* (Amsterdam: North-Holland Publishing Company, 1952), p. 332 and S. Kripke, 'Outline of a Theory of Truth', *Journal of Philosophy* 72 (1975), p. 700, n. 18.

> The first thing that comes to our mind when we hear them is that they are ordinary sophisms which are not worth refutation, hardly worth listening to them. ... However, it is indeed easier to discard them than to refute them definitively (*SW* v.18, p. 132; *LHP* v.1, p. 457).

Before Gödel, the average philosopher might well have nodded and passed on, still quite content to see only a 'shallow sophistry' in the paradoxes like that of the Liar. Once it is remarked that it was a *variation* of Eubulides' Liar which Gödel employed in his famous incompleteness theorem(s), then it is not unjust to observe that whether someone can only detect a shallow sophistry or a deep epistemological puzzle may well depend on depth of insight.

The antinomies that first prompted Kant to relate logic to metaphysics are not the modern logical paradoxes, although it might be noted that some early set theoretists have pointed out a similarity.[27] But insofar as modern logic can, via careful analogy, throw light on the philosophical question of the relation between logic, metaphysics and ontology, our point is that its discovery of the logical paradoxes, grounds the question so as to open, not close it.

Wood's appeal to Wittgenstein on the most promising way to deal with the logical paradoxes gives an idea of what is likely to be left of Hegel's insight regarding the epistemological significance of the antinomies when that is dealt with in the understanding's way:

> We might compare Hegel's treatment of philosophical paradoxes with the later Wittgenstein's. Wittgenstein held that contradictions or paradoxes do not "make our language less usable" because, once we "know our way about" and become clear about exactly where and why they arise, we can "seal them off"; we need not view a contradiction as "the local symptom of a sickness of the whole body." For Wittgenstein contradictions can be tolerated because they are marginal and we can keep them sequestered from the rest of our thinking; for Hegel, they arise systematically in the course of philosophical thought, but they do no harm so

[27] See e.g. W. Hessenberg, 'Grundbegriffe der Mengenlehre', *Abhandlungen der Fries'schen Schule*, Neue Folge 1,4 (1906), pp. 633 and 706; E. Zermelo in *Georg Cantor, Abhandlungen mathematischen und philosophischen Inhalts*, ed. by E. Zermelo (Hildesheim: Georg Olms, 1966), p. 377 (cited hereafter as 'Cantor').

> long as a system of speculative logic can keep them in their proper place ...[28]

Speculative logic as a special task force, keeping what is considered marginal sequestered from the rest, in its proper place? This wisdom of segregation in the guise of toleration at least suggests that how Hegel's *Logic* is read is not a scholastic issue. In this lies the importance of reading Hegel's *Logic as a logic*. It is logic that is haunted by antinomies. Letting logic off the hook in order to console common sense with Hegel's dialectic may work as an avoidance strategy for Hegelians. Claims such as

> none of Hegel's dialectic in the Logic is in opposition to 'ordinary logic'[29]

and

> Hegelian dialectic is no mysterious form of logic that transcends or is an alternative to ordinary logic.[30]

can survive because there is no sufficiently worked out theory in Hegel's logic, such as, for instance, a theory of arithmetic in the foundational studies of mathematics, that would defy categorical claims of this kind. But nor are such claims warranted. There are only some highly intriguing ideas in a highly difficult (abstract) realm of knowledge, formulated in no less difficult a language. In this situation, by *reversing the focus*, the discovery of the logical paradoxes can serve to open the question as to the nature of Hegel's dialectic. Hegel was not the one who ran unexpectedly into antinomies, it was Frege. In this sense, what is at stake now is higher order logic — not Hegel's idea of dialectic. Frege's logic has failed. The challenge is whether Hegel's idea of dialectic can make a point in the analysis of this failure. Higher order logic has the paradoxes and Hegel's idea of dialectic aims at making sense of contradictions in the enterprise of reason. Higher order logic with its paradoxes, undecidabilities, and incompleteness results is the touchstone of Hegel's idea of dialectic. *Hic Rhodus, hic saltus.*

In so far as Hegel's dialectic endorses a principle of freedom of concept formation (against Kant) it does challenge classical logic; unrestricted

[28] Wood, p. 3.

[29] Pinkard, 'Reply to Duquette', p. 22.

[30] Terry Pinkard, *Hegel's Dialectic: The Explanation of Possibility* (Philadelphia: Temple University Press, 1988), p. 5.

abstraction is incompatible with classical logic. Those who want to keep Hegel's dialectic in harmony with 'ordinary logic' will have to forgo an unlimited freedom of concept formation.[31] This is not to say that any of Hegel's actual concepts is indeed antinomical. What can be said safely is that Frege's *Grundgesetz* V (or unrestricted abstraction and extensionality) is in opposition to classical logic, in fact already unrestricted abstraction itself is in such opposition. All that is needed to make the link to Hegel is the realisation that unrestricted abstraction is in the spirit of Hegel's speculative philosophy.

To turn this point around: any claim that Hegel's dialectic is not in conflict with classical logic, can only succeed if Hegel can be shown to have proposed a restriction of concept formation to cope with Kantian antinomies. What Hegel does say with regard to Kant's Antinomy of Pure Reason is:

> The main point that has to be remarked is that the Antinomy is not just located in the four particular objects taken from Cosmology, but rather in *all* objects of all kinds, in all representations (*Vorstellungen*), notions, and Ideas. To know this and to recognize objects in this capacity (*Eigenschaft*) belongs to the essential of philosophical consideration; this capacity (*Eigenschaft*) accounts for what furthermore determines itself as the *dialectical* moment of the logical (*SW* v.8, § 48; *Enc*).

What is indeed lacking in Hegel is the actual production of an antinomy, such as the Liar, that would stand up to the standards of modern logic or, at least could be transformed into one. Accordingly the average Hegel scholar can say that, while Hegel may be seen as endorsing a principle which leads to antinomies, this does not mean that these antinomies are 'what he had in mind'. We do not and would not claim any such thing. In fact, we do not see the relevance of such a claim for the problem of a dialectical logic. There is more to that problem than scholastic rereading

[31] We take this to apply to Dieter Henrich's "substantivierte Aussageform" (propositional form turned noun, cf. the first section in his paper "Formen der Negation" in *Seminar: Dialektik in der Philosophie Hegels*, ed. by Rolf-Peter Horstmann (Frankfurt am Main: Suhrkamp, 1978), pp. 213–229) as well, although the endemic lack of precision in philosophical terminology does not allow the establishment of a conclusive link to unrestricted abstraction in logical terms.

of Hegel and while we do not dismiss ongoing efforts of Hegel interpretation, bringing Hegel into relation with modern logic requires competence in modern logic, a point that we find sorely neglected.[32] Unrestricted concept formation produces strange phenomena much stranger than 'table turning' ever was. And the second of the authors wants to add, that these phenomena are not even what he himself dreamt off, when he embarked on the project of making sense of Hegel's idea of dialectic in the framework of higher order logic some thirty-five years ago.

We do not want to close this section on "Hegel interpretation and logical illiteracy" without having produced at least one example of what we consider a fine alternative to a poor 'late twentieth-century perspective':

> Two of the greatest logico-mathematical discoveries of fairly recent times may in fact be cited as excellent and beautiful examples of Hegelian dialectic: I refer to Cantor's generation of transfinite numbers, and to Goedel's theorem concerning undecidable sentences. In the case of Cantor we first work out the logic of the indefinitely extending series of inductive, natural numbers, none of which transcends finitude or is the last in the series. We now pass to contemplate this series from without, as it were, and raise the new question as to how many of these finite, natural numbers we have. To answer this we must form the concept of the first transfinite number, the number which is the number *of* all these finite numbers, but is nowhere found *in* them or among then, which exists, to use Hegelian language, *an sich* in the inductive finite numbers, but becomes *für sich* only for higher-order comment. And Cantor's generation of the other transfinite numbers, into whose validity I shall not here enter, are all of exactly the same dialectical type. Goedel's theorem is also through and through dialectical, though not normally recognized as being so. It establishes in a mathematicized mirror of a certain

[32] It was not an analysis of Leukippos' and Demokritos' writings which substantiated any claim about atoms; it was not a *theory* of atoms that was handed down to us from the ancient Greeks, but an intriguing *idea*. Like every good idea there comes a time when one can do something with it. Hegel's idea of dialectic is just such a good idea to remember when confronted with the situation of higher order logic.

syntax-language that a sentence declaring itself, through a devious mathematicized circuit, to be unproveable in a certain language system, is itself unproveable in that system, thereby setting strange bounds to the power of logical analysis and transformation. But the unproveable sentence at the same time soars out of this logico-mathematical tangle since the proof of its unproveability in *one* language is itself a proof of the same sentence in *another* language of higher level, a situation than which it is not possible to imagine anything more Hegelian.[33]

3. Basic Ideas of Dialectical Logic

What we have said so far would remain as futile as any of those logically illiterate claims and polemics for or against a dialectical logic challenging classical logic, if we were not to give some indication as to what we propose as a dialectical logic, that is, a logic that does not require us "to frame determinations of things in terms of either/or propositions".[34] But we do not want to be misunderstood: the issue is too complex to be dealt with

[33] J. N. Findlay, 'The Contemporary Relevance of Hegel', in *Hegel. A Collection of Critical Essays*, ed. by A. MacIntyre (Notre Dame and London: University of Notre Dame Press, 1976), p. 6 f.

[34] By speaking of 'determinations of things in terms of either/or propositions' we mean determinations of things in terms of 'either x or not x', not any arbitrary x and y, i.e. 'either x or y', like, for instance: my computer is either made in Australia or standing on my desk. This remark is necessary in view of Pinkard's 'Reply to Duquette', (p. 20): "[Mr Duquette] says that ordinary logic requires us to frame determinations of things in terms of either/or propositions ... But logic *per se* does not require me to put things into either/or dichotomies; just note that the truth table for 'x or y' is different from the truth table for 'either x or y'." The truth table for 'either x or not x' is the same as that for 'x or not x'. Having said this, we hasten to emphasise that trivia of that kind are not the issue of the present section. What is the issue of the present section is that the identification of a logical constant with its truth table misses the point of an alternative logic altogether. In more technical terms, the message of the present section is that logic manifests itself in the so-called structural rules of a Gentzen-type formulation of logic. These structural rules regulate our dealing with assumptions, and this makes a difference to how the truth table of 'or', for instance, acts logically.

conclusively within the limited space of a paper of this kind; all we try to do is to evoke some interest and give some hints.[35]

Before turning to the more technical aspects, we want to try, at least, to convey some basic understanding of the issue in question. For that purpose, consider the following statements taken from different authors:

Hegel: To the ordinary (i.e. the sensuous-understanding) consciousness, the objects of which it knows count in their isolation for independent and resting on themselves.[36]

Cantor: [What we deal with in set theory are] manifolds of unconnected objects, i.e. manifolds of such a kind that removing any one or more of their elements has no influence on the remaining of the others.[37]

Wittgenstein: Each item can be the case or not the case while everything remains the same.[38]

Harris: The fundamental algebraical laws ... of commutation, association, and distribution ... hold only ... for entities that are externally related or are composed of externally related elements.[39]

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[35] Readers who want to know more regarding the mathematical logical side of what we propose as a dialectical logic are referred to: U. Petersen, 'Logic Without Contraction as Based on Inclusion and Unrestricted Abstraction', *Studia Logica* 64 (2000), pp. 365–403.

[36] *SW* v.8, § 45Z; *Enc.*

[37] Zermelo (ed.), 'Cantor', p. 470, n. 2; (our translation).

[38] Ludwig Wittgenstein, *Tractatus logico-philosophicus* (London: Routledge & Kegan Paul, 1969), p. 7.

[39] E. E. Harris, *Formal, Transcendental, and Dialectical Thinking* (Albany: State University of New York Press, 1987), pp. 32–33. This quotation is brutally edited to make it fit in with the other ones, although, we believe, it is not distorting. It is worthwhile, however, to quote a little more within the edited passage since it conveys, to our minds, an understanding of dialectical thinking that comes extremely close to our own. "If ... the units that made up a collection were internally related so that they affected one another in certain ways or constituted one another by their mutual relations, if, in short, we were dealing with wholes and not with mere collections, the order in which the elements were aggregated would not be indifferent and the algebraic laws would no longer hold." (*Ibid.* p. 33.) The emphasis, for us, lies on "the order ... would not be indifferent", and this is what we aim at by focusing on the structural rules below: roughly, the structural rules do for propositions (in logic) what the algebraic laws do for externally related objects, such as numbers (in arithmetic).

What shines through in these quotations, despite the differences in their claims, is an awareness of a possible alternative: are the objects that we are dealing with isolated things that have their properties independent of what anything else does around them, including our knowledge of them; or is ours a world of interconnectedness where it is in principle never possible to isolate an object, not even in thought?

This raises two questions. Firstly, why are the objects that we want to take into account in dialectical logic not severally independent? Differently put: what is there to relate entities internally, as distinct from externally? Secondly, how does classical logic have to be adjusted (if at all) in order to deal appropriately with objects which are internally related, or inherently connected?

Our answer to the first question in a nutshell: because conceptual thought is constitutive for all knowledge, and conceptual thought has the inescapable double character of form and content which manifests itself in an original ambiguity.[40]

This answer is derived from an analysis of Gödel's first incompleteness theorem, an analysis which cannot be presented here in full, though we shall try to give the gist of it.

Gödel's (formally) undecidable sentence involves a certain substitution function *sub* which satisfies the following condition

$$sub(\ulcorner\mathfrak{A}[x]\urcorner, n) = \ulcorner\mathfrak{A}[n]\urcorner\,,$$

where the little corners $\ulcorner\ \urcorner$ indicate the well-known device of numerical codification that Gödel introduced in his famous paper of 1931;[41] $\mathfrak{A}$ is a so-called nominal form,[42] a metatheoretical device for communicating that any well-formed expression of the language in question with certain indicated 'empty places' in which the expression in square brackets following it is to be inserted, may take its place; more intuitively, perhaps, any propositional form can be substituted for it. In plain words the above

[40] Dubbed *systemic ambiguity* by the second author. Cf. footnote 45 below.

[41] 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatshefte für Mathematik und Physik*, 38, pp. 173–198. Translated as 'On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems' in J. van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic* (Cambridge, Massachusetts: Harvard University Press, 1967), pp. 596–616.

[42] Cf. Schütte, *Proof Theory* (Berlin, Heidelberg, New York: Springer-Verlag), p. 11.

equation reads: $sub(\ulcorner\mathfrak{A}[x]\urcorner, n)$ equals the Gödel number of the result of replacing every indicated occurrence of x in $\mathfrak{A}[x]$ by the numeral n. Gödel's trick consists in taking for both arguments of this substitution function the Gödel number of the expression $\mathfrak{A}[sub(x,x)]$, i.e. $\ulcorner\mathfrak{A}[sub(x,x)]\urcorner$. Let us take $\mathbf{k}_{\mathfrak{A}}$ as an abbreviation for $\ulcorner\mathfrak{A}[sub(x,x)]\urcorner$ and we obtain the following (indirect) 'fixed point property':

$$sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}}) = \ulcorner\mathfrak{A}[sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}})]\urcorner .$$

The reason that this is called a "fixed point property" should become sufficiently clear when we take the abbreviation $f_{\mathfrak{A}}$ for $sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}})$:

$$f_{\mathfrak{A}} = \ulcorner\mathfrak{A}[f_{\mathfrak{A}}]\urcorner$$

and call $f_{\mathfrak{A}}$ a fixed point with regard to $\mathfrak{A}$: if $\mathfrak{A}$ is regarded as a propositional function, then its value for the argument $f_{\mathfrak{A}}$ is $f_{\mathfrak{A}}$ itself. Such a fixed point property causes trouble for the expressibility of basic semantical concepts on the level of the formalised theory itself (i.e. as an arithmetical predicate, such as, for instance, the predicate of being a prime number), most notably that of truth, i.e. a predicate that satisfies the following 'truth condition':

$$tru(\ulcorner A\urcorner) \leftrightarrow A .$$

To see this, assume the existence of such a predicate *tru*. Obviously it satisfies

$$tru(f) \leftrightarrow tru(f) ,$$

and by the above fixed point property there is a fixed point $f_{\neg tru}$ such that:

$$f_{\neg tru} = \ulcorner\neg tru(f_{\neg tru})\urcorner .$$

By the substitutivity of equal numbers in arithmetic propositions these two yield:

$$tru(f_{\neg tru}) \leftrightarrow tru(\ulcorner\neg tru(f_{\neg tru})\urcorner) .$$

On the other hand, by the above truth condition, one has

$$tru(\ulcorner\neg tru(f_{\neg tru})\urcorner) \leftrightarrow \neg tru(f_{\neg tru}) .$$

By the transitivity of $\leftrightarrow$, the last two yield:

$$tru(f_{\neg tru}) \leftrightarrow \neg tru(f_{\neg tru}) ,$$

i.e. an antinomy.[43]

What happens — in the establishment of the (indirect) fixed point property which lies at the bottom of these results — is that we have (the formal representative of) a number here, which we called $\mathbf{k}_{\mathfrak{A}}$, which occurs as the argument of the function *sub* in two different roles. One time it occurs as an innocent number, i.e. it is being constructed from 0 in a series of steps of adding 1. The other time, however, it occurs as a hieroglyphic behind which a complex proposition is hiding. The substitution function juggles with these two sides of $\mathbf{k}_{\mathfrak{A}}$, which accounts for the curious double character in the employment of $sub(\mathbf{k}_{\mathfrak{A}}, \mathbf{k}_{\mathfrak{A}})$, and according to which way we look at this number, we get conflicting results. This is what we take as our paradigm of a conflict between form and content.

In other words, Gödel's construction of a formally undecidable sentence involves a mathematically immaculate form of a use-mention confusion.[44] This confusion is the source of a certain ambiguity which is inescapable once a sufficient amount of arithmetic is available. It provides the answer to our first question. Differently put: the understanding's way, governed by the silent assumption that the objects of our thought can be treated as severally independent, unconnected, externally related, is incompatible with the actual existence of a connection, an internal relation, provided by Gödel's encoding.[45]

This confusion does no harm, as long as there are no semantical concepts available which would be sufficient to establish a connection between the formal system and its intended interpretation, like that of truth or

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[43] Readers who 'find themselves puzzled' in some of the logical moves involved in this reasoning may find it comforting to know that the technicalities do indeed require some basic skill in mathematical logic, in the absence of which the correctness of these moves would have to be taken on trust. We refer to our footnote 9 above. Readers with more serious ambitions might find it helpful to consult a survey article such as C. Smoryński, 'The Incompleteness Theorems', *Handbook of Mathematical Logic*, ed. by J. Barwise (Amsterdam: North-Holland Publishing Company, 1977), in particular, pp. 826–7. A condensed treatment can also be found in G. Takeuti, *Proof Theory* (Amsterdam: North-Holland Publishing Company, 1987), in particular, pp. 82–85.

[44] R. L. Goodstein, *Essays in the Philosophy of Mathematics* (Leicester: Leicester University Press, 1967), p. 20: "The code has been used and mentioned, and there is no self-reference."

[45] In U. Petersen, *Diagonal Method and Diagonal Logic* (Osnabrück: Der Andere Verlag, 2002), section 111d, p. 1530, the label "systemic ambiguity" is introduced for this phenomenon.

satisfaction, for instance. It is only the source of incompleteness and undecidability results. One small step, however, and hell breaks loose: add a sentence which is provable in a meta-theory, like that of the consistency of the object-theory in question, and everything becomes provable. The classic example is that of a reflection principle for the provability predicate of first order arithmetic, provable in second order arithmetic,[46] but incompatible within first order arithmetic itself. Such is the situation of theories based on classical logic, in which a certain amount of arithmetic is available.

We thus come to our second question: how can we take account of the internal relatedness of our objects? Differently put: how can we avoid the implicit assumption of the understanding's way that objects are severally independent? How does an assumption of several independency manifest itself on the logical level? Is logical reasoning possible without the assumption that the objects of our thought are severally independent?

This is a tricky question, or rather cluster of questions, because it more or less implicitly requires an answer to the question: what is logic? Or, at least, what is the difference between classical and non-classical logics?

Our answer to this question is derived from some well-established techniques within proof theory, a familiarity which, unfortunately is hardly to be found amongst philosophers in the Hegelian tradition, and only little more amongst philosophers in the analytic tradition. These techniques are linked to the name of Gerhard Gentzen and their central features are cut elimination and normalisation.

In 1934, Gentzen proposed a formulation of classical and intuitionistic logic in terms of so-called *sequents* ("Sequenzen").[47] We shall restrict our attention here to the case of intuitionistic logic, since it is slightly simpler to present while it shows, at the same time, all the relevant features required to make our point.

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[46] This simply says: if $\ulcorner A \urcorner$ is the Gödel number of a provable formula A, then A; less technical: if A is provable, then A.

[47] Gerhard Gentzen, 'Untersuchungen über das logische Schließen', *Mathematische Zeitschrift*, 41 (1934), pp. 176–210 and 405–431. Translated by M.E. Szabo in *The Collected Papers of Gerhard Gentzen* (Amsterdam and London: North-Holland Publishing Company, 1969).

A sequent has the following form

$$A_1, \ldots, A_n \Rightarrow C \,,$$

where $A_1, \ldots, A_n, C$ are formulas. The formulas left of $\Rightarrow$ are considered assumptions, the formula on the right of $\Rightarrow$ the hypothesis. Rules in Gentzen's formulation of logic are divided into two kinds: structural rules and operational rules. The rules for handling logical constants are the operational rules. In the case of "or", in symbols $\vee$, they look like this (where Γ and Π denote sequences, as distinct from sequents, of formulas, such as $A_1, \ldots, A_n$, for instance):
Introduction left:

$$\frac{A, \Gamma \Rightarrow C \qquad\qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \,.$$

Introduction right:

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \text{and} \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \,.$$

These rules perfectly mirror the truth values if one takes a sequent to be true if one of the assumptions is false, or the hypothesis is true. They do not, however, fully determine the meaning (or behaviour) of the disjunction "or". What is needed in addition are rules which regulate the handling of the assumptions:
Weakening

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C} \,.$$

Exchange

$$\frac{\Gamma, A, B, \Rightarrow C}{\Gamma, B, A, \Rightarrow C} \,.$$

Contraction

$$\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C} \,.$$

In words: weakening says that assumptions may be added according to taste, exchange says that the order of two assumptions may be reversed, and contraction says that having an assumption once is as good as having it twice, or as Girard put it:

> *contraction* is the fingernail of infinity in propositional calculus: it says that what you have, you will always keep, no matter how you use it.[48]

Note that these structural rules involve no logical constants. Nevertheless, they are the true backbone of classical logic. As Girard put it:

> these rules are the most important of the whole calculus, for, without having written a single logical symbol, we have practically determined the future behaviour of the logical operations.[49]

And:

> It is not too excessive to say that a logic is essentially a set of structural rules![50]

One example in which the future behaviour of the logical operation $\vee$ ('or') is determined by the structural rules is *tertium non datur*, $A \vee \neg A$. Without contraction it is impossible to obtain *tertium non datur* from the above operational rules for $\vee$.

In the light of these considerations regarding the role of assumptions in logic, we can now formulate our answer to the second question: because of the double character of concepts, two occurrences of the same statement in a proof may not without further provision be assumed to have the same truth-value, i.e. we look at formulas in logic as *tokens* and not types. This view of formulas as tokens can be incorporated in Gentzen's formulation of logic by dropping the rule which allows the reproduction of assumptions *ad libitum*: contraction.[51] This idea was put forward in 1980 (by the second author):

> Having inferred B from A and A $\rightarrow$ B we cannot expect ... that A and A $\rightarrow$ B are still available as presuppositions (assumptions). It is possible that they have changed in the process of inferring,

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[48] J.-Y. Girard, 'Towards a Geometry of Interaction', *Contemporary Mathematics*, 92 (1989), pp. 69–108 at p. 78.

[49] J.-Y. Girard, Y. Lafont, P. Taylor, *Proofs and Types* (Cambridge: Cambridge University Press, 1989), p. 30.

[50] J.-Y. Girard, 'Towards a Geometry of Interaction', p. 78.

[51] This is not to be confused with *adding* assumption; that's what weakening does. Contraction allows assumptions to be used more than once and in that sense it allows the reproduction of assumptions; or, if you prefer: multiplication of resources at no extra costs.

> that they have been exhausted, so to speak. This means we interpret the implication A → B as "A transfers into B". In this way we want to take account of the peculiarity of unrestricted abstraction.[52]

In this sense, dialectical logic is a resource conscious logic,[53] a logic in which attention is paid to the manipulation of assumptions. Classical logic has no space for a dynamics of assumptions: the structural rules override it; truth and falsity is determined before we start reasoning. Reasoning under the rule of classical logic can only establish truth *for us*; it is subjective in the sense that the objective state of affairs is determined before we start reasoning. Classical logic cannot allow reasoning to be part of the truth, and in so far as the paradigm of classical logic is the understanding's world, truth can never reside in thought determinations.[54] Classical logic has no truth within itself; it can only be truth preserving, never generating.[55]

4. Dialectical Thought versus Finite Thought — the Example of the Complement

Having fixed a logic which does not succumb to either-or reasoning in the specific sense that unrestricted abstraction is allowed without causing 'head-on contradictions' ("kontradiktorische Widersprüche"), there is still the question of what that actually means for logical reasoning.

It will perhaps be clear that the difference between dialectical thought and classical thought is subtle and just as the structure of the cell does

[52] U. Petersen, *Die logische Grundlegung der Dialektik* (München: Wilhelm Fink Verlag, 1980), p. 97; (our translation).

[53] The term is taken from A.S. Troelstra, *Lectures on Linear Logic* (Stanford: Center for Studies of Language and Information, 1992), p. 1. In the past ten years one particular specimen of a resource conscious logic has had a major impact on computer science, the linear logic of J.-Y. Girard.

[54] "The question regarding the truth of the thought determinations must seem strange to the ordinary consciousness ... This question, however, is just what matters (*worauf es ankömmt*)" (Hegel *SW* v. 8, § 24Z(2); *Enc.*).

[55] This has to be contrasted with the following: "Hegel was also worried about logic's formality, since he thought it doubtful that logic could be 'true' if it were purely formal. He could have avoided that worry altogether if he had been in the position to hold the contemporary view that logic is not intended to *provide* truth at all but just to *preserve* it." Pinkard, 'Reply to Duquette' at p. 23.

not reveal itself to the naked eye, the subtleties of dialectical thought do not reveal themselves to plain thinking. The most spectacular aspect of unrestricted abstraction is a so-called (direct) *fixed point property.*[56] What it says in plain words is that to every concept, the list $\mathfrak{F}$ of properties of which contains occurrences of y, there is an object f, the *fixed point* of $\mathfrak{F}$, such that a replacement of these occurrences of y by occurrences of f results in a concept which equals f. Since this will make the head of a logician go into a spin, we add a formulation in the artificial language of symbolic logic:

$$\lambda x\, \mathfrak{F}[x, f] = f\ .$$

This gives rise to a beautiful example of a theorem in classical logic which no longer prevails in its original form in dialectical logic (as outlined above). It can be found in Leibniz in the following form (including a proof):

> Theor. X.
>
> *Detractum et Residuum sunt incommunicantia.*
>
> Si L – A∞N, dico A et N nihil habere commune. Nam ex definitione detracti et Residui omnia quae sunt in L manent in N praeter ea quae sunt in A, quorum nihil mane in N.[57]

In modern set theory it runs (without a proof)

A set and its complement are disjunct.

In set theoretical symbolism:

$$\mathrm{M} \cap \complement(\mathrm{M}) = \varnothing\ ,$$

where $\complement$(M) is the complement of M and ∅ is the empty set. In other words: the intersection between a set and its complement is empty. Or: M et $\complement$(M) nihil habere commune.

This touches on an extremely delicate and crucial point. Is it possible, in principle, to divide the world into two disjunct parts, the union of which is the world, i.e. is it possible to have a division of the world

---

[56] This is to be distinguished from the indirect fixed point property from p. 38, insofar as the fixed point is not hidden within the little corners ⌜ ⌝. Labelling fixed points 'direct' and 'indirect' is not common in logic; it suggests itself for logicians who want to accommodate for unrestricted abstraction.

[57] G. W. Leibniz, *Schriften zur Logik und zur philosophischen Grundlegung von Mathematik und Naturwissenschaft*, ed. by Herbert Herring (Frankfurt am Main: Suhrkamp, 1996), p. 170.

without remainder? The classical logician has provided an answer before the philosopher comes on the scene: *tertium non datur* does just that.

If the classical logician is right, there is no room for Hegel's dialectic. All that might be possible is a diluted form like a hermeneutics of categories. But then, if the classical logician is right, there is also no room for unrestricted abstraction, because unrestricted abstraction (with some basic logic) provides the (direct) fixed point property. And what the (direct) fixed point property for terms tells us is that there is an element f (a 'fixed point') such that $\mathsf{C}(f) = f$. This has a decisive impact on the above theorem: on the one hand, we have

$$f \cap \mathsf{C}(f) = \varnothing$$

by the theorem, and on the other hand

$$\mathsf{C}(f) = f$$

by the fixed point property, i.e.

$$f \cap f = \varnothing$$

by substitutivity of equals. In words: the intersection of f with itself is empty. In classical set theory this means that $f = \varnothing$, i.e. f itself is empty; but then, the complement of the empty set is the universal set. From a classical position this leaves no choice but to exclude the fixed point f as unpalatable. This is what logicians have mostly done since Russell's discovery of his antinomy. The decision that weird terms such as Russell's class have to be avoided has been handed down to philosophers of somewhat Hegelian persuasion. But when modern logic finally arrives at the level of philosophers it has been reduced to a heap of dead bones not much different in character to those that Hegel saw in the logic of his time.

So what is wrong in Leibniz' reasoning, or the reasoning of modern set theory, from a dialectical point of view? The answer is that it does not take into account the role of assumptions in the reasoning related to notions of 'incommunicantia' or 'disjunct'; more specifically to the notion of 'and' that is involved in these concepts.

In the absence of contraction the classical truth tables for conjunction do not fully determine just one particular notion of conjunction. As a consequence, dialectical logic distinguishes two forms of intersection: $\cap$ and $\sqcap$; relying on the two different notions of conjunction. Both notions of conjunction are characterised by the same (classical) truth values. What distinguishes them is the handling of assumptions.

Between them the two notions of conjunction divide all the properties that their classical counterpart combines in one. Leibniz' theorem, for instance, does indeed hold for the one form of intersection, communicated by $\sqcap$:

$$\mathrm{M} \sqcap \complement(\mathrm{M}) = \varnothing\,;$$

but what fails is $f \sqcap f = f$. For the other form of intersection, communicated by $\cap$ the situation is exactly the other way round.

This situation gives rise to a variation on an eminently Hegelian theme, the identity and non-identity of being and nothing. What can be established with the help of the fixed point property is that to every concept there exists another one, a 'doppelgänger' as it were, which is equal but not identical to the original one, i.e. any object that falls under one of them also falls under the other. Still, they are not the same in the following sense: in so far as they may be regarded as objects themselves, they have different properties, i.e. they cannot be substituted for each other regardless of context.

In Hegel's (translated) words:

> Their difference is ... completely empty ...; it thus does not subsist in themselves, but only in a third, in *opinion* (*SW* v.4, p. 101; *SL*, p. 92).

Contraction free logic with unrestricted abstraction has space for a phenomenon of this kind; in fact, it creates such phenomena in abundance. They are the mysteries of the understanding, and their presence calls for another sacrifice on the part of the classical doctrine: 'extensionality' is just the principle that if two concepts subsume the same objects under them, then they may be substituted for each other *salva veritate*.[58] This principle of identity, an integral part of Frege's logic in the *Grundgesetze*, is incompatible with the possibility of unrestricted abstraction in higher order logic. This is the more remarkable as Frege's celebrated distinction

[58] Thus Leibniz defined: "*Eadem* sunt quorum unum potest substitui alteri salva veritate." G. W. Leibniz, *Schriften zur Logik und zur philosophischen Grundlegung von Mathematik und Naturwissenschaft*, p. 156. ("Those terms are 'the same' of which one can be substituted for the other without loss of truth." *Leibniz. Logical Papers. A Selection*, ed. and trans. by G. H. R. Parkinson (Oxford: Clarendon Press, 1966), p. 123).

of sense and reference was, and that not in the last instance, meant to provide support for extensionality, at least in logic and arithmetic.[59] To paraphrase Hegel:

> There is mystery in higher order logic, only however for the understanding which is ruled by the principle of abstract identity.

Or, as someone by no means less famous than Hegel has not quite said some time before Hegel:

> There are more things in higher order logic,
> Than are dreamt of in understanding's philosophy.

---

[59] "I use the word "equal" to mean the same as "coinciding with" or "identical with"; and the sign of equality is actually used in arithmetic in this way. The opposition that may arise against this will very likely rest on an inadequate distinction between sign and thing signified." *Gottlob Frege — The Basic Laws of Arithmetic. Exposition of the System*, trans. by M. Furth (Berkeley, Los Angeles: University of California Press, 1964), p. 6.

Some Additions and Corrections to *Diagonal Method and Dialectical Logic*[1]

Uwe Petersen

The following additions are meant to indicate some of the directions my research has taken since the publication of [**15**].

1. Addition 124g. Interpreting Weakening in $\mathbf{LB}^{\circ}$

The point of this addition is to show that sacrificing weakening does not restrict expressive power in the presence of unrestricted abstraction.

A central issue in the development of a speculative logic is the question of how far one gets without any structural rules. In this context I shall present an interpretation of the formalized theory $\mathbf{LD}_{\lambda}$ as presented in [**15**], p. 472, definition 41.22 (4) (essentially Gentzen's **LK** without contraction but equipped with unrestricted λ-abstraction) in its intuitionistic linear subsystem. The relevant point is that $\perp \rightarrow A$ is available due to the definition of $\perp$ by means of unrestricted abstraction. The result is not in any way surprising but it seems to me of interest in view of linear logic and also in view of my ambitions to build logic without any structural rules.

The principal approach goes back to [**10**], but [**7**] was to become more influential. The approach taken here is in character closer to [**17**], pp. 49 f, although it still differs from it, not only in that I use different primitive symbols. It should be clear, however, that the present approach is in no way original and that it can be extended to theories built on linear logic, *i.e.*, abandoning weakening is in character very similar to shifting to intuitionistic logic from classical logic: in both cases it is double negation which holds the key to the interpretation, in the sense that adding double negation yields classical logic.

[1] Item [**15**] in the references for this paper starting on p. 90.

I begin by providing the relevant definitions.

DEFINITION 1.1. The formalized theory $\mathbf{LB}^\circ$ is obtained from the formalized theory $\mathbf{LB}^+$ introduced in [**15**], p. 1682, definition 124.6 (4), by dropping weakening.

INTUITIVE CONSIDERATION 1.2. The notion $\bot$ of *falsum* provides for the deducibility of $\bot \Rightarrow A$ (122.46v in [**15**], p. 1663). This, in turn, provides for a substitution of weakening: instead of $A \to (B \to A)$ the following is $\mathbf{LB}^\circ$-deducible:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \bot \Rightarrow \neg B}{\neg A, A \Rightarrow \neg B}}{\neg\neg B, \neg A, A \Rightarrow \bot}}{B, \neg A \Rightarrow \neg A} \qquad \bot \Rightarrow \bot}{\neg\neg A, \neg\neg B, \neg A \Rightarrow \bot}}{\neg\neg A, \neg\neg B \Rightarrow \neg\neg A}$$

Obviously $A \Rightarrow \neg\neg A$ is $\mathbf{LB}^\circ$-deducible. If double negation $\neg\neg A \Rightarrow A$ were also available, then this would be sufficient to prove weakening in the form $A \to (B \to A)$:

$$\begin{array}{c} \dfrac{\dfrac{B \Rightarrow \neg\neg B \qquad \dfrac{A \Rightarrow \neg\neg A \qquad \neg\neg A, \neg\neg B \Rightarrow \neg\neg A}{A, \neg\neg B \Rightarrow \neg\neg A}}{A, B \Rightarrow \neg\neg A} \qquad \neg\neg A \Rightarrow A}{A, B \Rightarrow A} \\ \hline\hline \Rightarrow A \to (B \to A) \end{array}$$

Apparently, however, weakening right is needed in a $\mathbf{LB}^\circ$-deduction of double negation:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, \bot}}{\Rightarrow A, A \to \bot} \qquad \bot \Rightarrow \bot}{(A \to \bot) \to \bot \Rightarrow A}$$

This is why I make recourse to the kind of interpretation that Gödel employed for the purpose of interpreting classical logic within intuitionistic logic.

PROPOSITION 1.3. *Inferences according to the following schemata are* **LB**°*-derivable.*

(1.3i)
$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \neg\neg A}$$

(1.3ii)
$$\frac{\Gamma \Rightarrow \neg\neg A}{\Gamma, \neg A \Rightarrow C}$$

(1.3iii)
$$\frac{A, \Gamma \Rightarrow \bot}{\neg\neg A, \Gamma \Rightarrow C}$$

(1.3iv)
$$\frac{\neg\neg A, \Gamma \Rightarrow \bot}{A, \Gamma \Rightarrow \bot}$$

Proof. Straightforward. I only show 1.3ii as an example. Employ 122.46v feom [**15**], p. 1663:

$$\frac{\Gamma \Rightarrow \neg\neg A \qquad \dfrac{\neg A \Rightarrow \neg A \qquad \bot \Rightarrow C}{\neg A, \neg\neg A \Rightarrow C}}{\Gamma, \neg A \Rightarrow C}\clubsuit$$

QED

DEFINITION 1.4. $\|X\|$ is defined inductively as follows:

(1) $\|u\| :\equiv u$, u being a free or bound variable;
(2) $\|s \sqsubseteq t\| :\equiv \neg\neg(\|s\| \sqsubseteq \|t\|)$;
(3) $\|\lambda x\, \mathfrak{F}[x]\| :\equiv \lambda x \|\mathfrak{F}[x]\|$;
(4) If $\mathit{\Gamma}$ ist the sequence $A_1, \ldots, A_{\mathrm{m}}$, then $\|\mathit{\Gamma}\|$ is the sequent $\|A_1\|, \ldots, \|A_{\mathrm{m}}\|$;
(5) $\|\mathit{\Gamma} \Rightarrow C\| :\equiv \|\mathit{\Gamma}\| \Rightarrow \|C\|$.

PROPOSITION 1.5. $\|C\|$ *has the form* $\neg\neg A$.

Proof. This is an obvious consequence of clause (2) of the foregoing definition in view of the fact that the outermost symbol of every wff in the language of **LB**° is $\sqsubseteq$: If $C \equiv s \sqsubseteq t$, then $\|C\| \equiv \neg\neg(\|s\| \sqsubseteq \|t\|)$. QED

Proposition 1.6. *Sequents according to the following schemata are* $\mathbf{LB}^{\circ}$*-deducible.*

(1.6i) $\|\bot\| \Rightarrow \bot$

(1.6ii) $\neg\neg\|A\| \Rightarrow \|A\|$

(1.6iii) $\|(A \to \bot) \to \bot\| \Rightarrow \|A\|$

(1.6iv) $\|\bot\|, \neg B \Rightarrow \bot$

(1.6v) $\|A\|, \|B\| \Rightarrow \|A\|$

(1.6vi) $\neg\neg(s \in \|b\|) \Rightarrow s \in \|b\|$

(1.6vii) $\neg\neg(s \in \lambda x \, \|\mathfrak{A}[x]\|) \Rightarrow s \in \lambda x \, \|\mathfrak{A}[x]\|$

Proof. Re 1.6i.

$$\dfrac{\dfrac{\dfrac{\bot \Rightarrow \bot}{\Rightarrow \lambda\bot \sqsubseteq \lambda\bot}}{\Rightarrow \neg\neg(\lambda\bot \sqsubseteq \lambda\bot)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \sqsubseteq a \Rightarrow a \sqsubseteq a}{\Rightarrow \lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))}}{\Rightarrow \neg\neg(\lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))} \qquad \bot \Rightarrow \bot}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot \Rightarrow \bot}}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot) \Rightarrow \bot}}{\dfrac{\dfrac{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda y\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x) \Rightarrow \bot}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda x\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x)) \Rightarrow \bot}}{\|\mathcal{V} \sqsubseteq \lambda x\,(\mathcal{V} \sqsubseteq x)\| \Rightarrow \bot}}$$

Re 1.6ii. Let $\neg\neg\, A_1 \equiv \|A\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\|A\| \Rightarrow \|A\|}{\|A\| \Rightarrow \neg\neg A_1} \qquad \dfrac{\neg A_1 \Rightarrow \neg A_1 \qquad \bot \Rightarrow \bot}{\neg\neg A_1, \neg A_1 \Rightarrow \bot}}{\dfrac{\dfrac{\dfrac{\|A\|, \neg A_1 \Rightarrow \bot}{\neg\neg\|A\|, \neg A_1 \Rightarrow \bot}}{\neg\neg\|A\| \Rightarrow \neg\neg A_1}}{\neg\neg\|A\| \Rightarrow \|A\|}}$$

Re 1.6iii. Let $\neg\neg\, A_1 \equiv \|A\|$ according to proposition 1.5. Employ 122.46v from [**15**], p. 1663, and 1.6i:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A_1 \Rightarrow \neg A_1 \qquad \bot \Rightarrow \|\bot\|}{\neg A_1, \neg\neg A_1 \Rightarrow \|\bot\|}}{\neg A_1, \|A\| \Rightarrow \|\bot\|}}{\neg A_1 \Rightarrow \lambda\|A\| \sqsubseteq \lambda\|\bot\|}}{\neg A_1 \Rightarrow \neg\neg(\lambda\|A\| \sqsubseteq \lambda\|\bot\|)}}{\neg A_1 \Rightarrow \|(A \to \bot)\|} \qquad \|\bot\| \Rightarrow \bot}{\neg A_1, \lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\| \Rightarrow \bot}}{\neg A_1 \Rightarrow \neg(\lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\|)} \qquad \bot \Rightarrow \bot}{\neg\neg(\lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\|), \neg A_1 \Rightarrow \bot}}{\neg\neg(\lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\|) \Rightarrow \neg\neg A_1}}{\|(A \to \bot) \to \bot\| \Rightarrow \|A\|}$$

Re 1.6iv. Employ 122.46v from [**15**], p. 1663:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \Rightarrow \bot}{\Rightarrow \lambda\bot \sqsubseteq \lambda\bot}}{\Rightarrow \neg\neg(\lambda\bot \sqsubseteq \lambda\bot)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \sqsubseteq a \Rightarrow a \sqsubseteq a}{\Rightarrow \lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))}}{\Rightarrow \neg\neg(\lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))} \qquad \dfrac{\bot \Rightarrow B \qquad \bot \Rightarrow \bot}{\bot, \neg B \Rightarrow \bot}}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot, \neg B \Rightarrow \bot}}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot), \neg B \Rightarrow \bot}}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda x\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x), \neg B \Rightarrow \bot}}{\dfrac{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda x\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x)), \neg B \Rightarrow \bot}{\|\bot\|, \neg B \Rightarrow \bot}}$$

Re 1.6v. Let $\neg\neg\, A_1 \equiv \|A\|$ and $\neg\neg\, B_1 \equiv \|B\|$ according to proposition 1.5. Employ 122.46v from [**15**], p. 1663:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1 \Rightarrow A_1 \qquad \bot \Rightarrow \neg B_1}{\neg A_1, A_1 \Rightarrow \neg B_1}}{\neg\neg B_1, \neg A_1, A_1 \Rightarrow \bot}}{\neg\neg B_1, \neg A_1 \Rightarrow \neg A_1} \qquad \bot \Rightarrow \bot}{\neg\neg A_1, \neg\neg B_1, \neg A_1 \Rightarrow \bot}}{\neg\neg A_1, \neg\neg B_1 \Rightarrow \neg\neg A_1}}{\|A\|, \|B\| \Rightarrow \|A\|}$$

Re 1.6vi.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(s\in b) \Rightarrow \neg(s\in b) \qquad \bot \Rightarrow \bot}{\neg\neg(s\in b), \neg(s\in b) \Rightarrow \bot}}{s\in\lambda x\,\neg\neg(x\in b), \neg(s\in b) \Rightarrow \bot}\;1.3\text{iii}}{\neg\neg(s\in\lambda x\,\neg\neg(x\in b)), \neg(s\in b) \Rightarrow \bot}}{\neg\neg(s\in\lambda x\,\neg\neg(x\in b)) \Rightarrow \neg\neg(s\in b)}}{\neg\neg(s\in\lambda x\,\neg\neg(x\in b)) \Rightarrow s\in\lambda x\,\neg\neg(x\in b)}$$

Re 1.6vii. Let $\neg\neg\,\mathfrak{A}_1[s] \equiv \|\mathfrak{A}[s]\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\mathfrak{A}_1[s] \Rightarrow \neg\mathfrak{A}_1[s]}{\neg\neg\mathfrak{A}_1[s], \neg\mathfrak{A}_1[s] \Rightarrow \bot}}{\|\mathfrak{A}[s]\|, \neg\mathfrak{A}_1[s] \Rightarrow \bot}}{s\in\lambda x\,\|\mathfrak{A}[x]\|, \neg\,\mathfrak{A}_1[s] \Rightarrow \bot}\;1.3\text{iii}}{\neg\neg(s\in\lambda x\,\|\mathfrak{A}[x]\|), \neg\,\mathfrak{A}_1[s] \Rightarrow \bot}}{\neg\neg(s\in\lambda x\,\|\mathfrak{A}[x]\|) \Rightarrow \neg\neg\,\mathfrak{A}_1[s]}}{\neg\neg(s\in\lambda x\,\|\mathfrak{A}[x]\|) \Rightarrow \|\mathfrak{A}[s]\|}}{\neg\neg(s\in\lambda x\,\|\mathfrak{A}[x]\|) \Rightarrow s\in\lambda x\,\|\mathfrak{A}[x]\|}$$

QED

PROPOSITION 1.7. *Inferences according to the following schemata are* **LB**°*-derivable.*

(1.7i) $$\dfrac{\Gamma \Rightarrow \neg\neg(s\in\|t\|)}{\Gamma \Rightarrow s\in\|t\|}$$

(1.7ii) $$\frac{s \in \|t\|, \Gamma \Rightarrow C}{\neg\neg(s \in \|t\|), \Gamma \Rightarrow C}$$

Proof. This are fairly immediate consequence of 1.6vi and 1.6vii. QED

The next step is to show that the interpretation of every $\mathbf{LD}_\lambda$-derivable inference is $\mathbf{LB}^\circ$-derivable.

PROPOSITION 1.8. *Inferences according to the following schemata are* $\mathbf{LB}^\circ$*-derivable.*

(1.8i) $$\frac{\|\Gamma \Rightarrow C\|}{\|A, \Gamma \Rightarrow C\|}$$

(1.8ii) $$\frac{\|\Gamma, A, B, \Pi \Rightarrow C\|}{\|\Gamma, B, A, \Pi \Rightarrow C\|}$$

(1.8iii) $$\frac{\|\Gamma \Rightarrow A\| \qquad \|A, \Pi \Rightarrow C\|}{\|\Gamma, \Pi \Rightarrow C\|}$$

(1.8iv) $$\frac{\|\Gamma \Rightarrow \mathfrak{A}[s]\| \qquad \|\mathfrak{B}[s], \Pi \Rightarrow C\|}{\|\lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y], \Gamma, \Pi \Rightarrow C\|}$$

(1.8v) $$\frac{\|\Gamma, \mathfrak{A}[a] \Rightarrow \mathfrak{B}[a]\|}{\|\Gamma \Rightarrow \lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y]\|}$$

Proof. Re 1.8i. This is 'weakening'. Employ 1.6iv. Distinguish two cases: empty antecedent or not. In the first case, let $\neg\neg B_1 \equiv \|B\|$ and $\neg\neg C_1 \equiv \|C\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\Rightarrow \|C\|}{\Rightarrow \neg\neg C_1} \qquad \dfrac{\neg C_1 \Rightarrow \neg C_1 \qquad \bot \Rightarrow \neg B_1}{\neg C_1, \neg\neg C_1 \Rightarrow \neg B_1}}{\dfrac{\dfrac{\dfrac{\neg C_1 \Rightarrow \neg B_1}{\neg\neg B_1 \Rightarrow \neg\neg C_1}}{\|B\| \Rightarrow \|C\|}}{\|B \Rightarrow C\|}}$$

In the second case, let Γ be the sequence $A_1, \dots, A_{\mathrm{m}}$. Employ 1.6iv.

$$\dfrac{\|A\|, \|A_1\| \Rightarrow \|A_1\| \qquad \dfrac{\|A_1, \dots, A_m \Rightarrow C\|}{\|A_1\|, \dots, \|A_m\| \Rightarrow \|C\|}}{\dfrac{\|A\|, \|A_1\|, \dots, \|A_m\| \Rightarrow \|C\|}{\|A, A_1, \dots, A_m \Rightarrow C\|}} \clubsuit$$

Re 1.8ii. This is 'exchange'. Obvious. left to the reader.
Re 1.8iii. This is 'cut'.

$$\dfrac{\dfrac{\|\Gamma \Rightarrow A\|}{\|\Gamma\| \Rightarrow \|A\|} \qquad \dfrac{\|A, \Pi \Rightarrow C\|}{\|A\|, \|\Pi\| \Rightarrow \|C\|}}{\dfrac{\|\Gamma\|, \|\Pi\| \Rightarrow \|C\|}{\dfrac{\|\Gamma\|, \|\Pi\| \Rightarrow \|C\|}{\|\Gamma, \Pi \Rightarrow C\|}}} \clubsuit$$

Re 1.8iv. This is $\sqsubseteq$-left rule. Let $\neg\neg C_1 \equiv \|C\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\dfrac{\|\Gamma \Rightarrow \mathfrak{A}[s]\|}{\|\Gamma\| \Rightarrow \|\mathfrak{A}[s]\|} \qquad \dfrac{\dfrac{\|\mathfrak{B}[s], \Pi \Rightarrow C\|}{\|\mathfrak{B}[s]\|, \|\Pi\| \Rightarrow \|C\|}}{\|\mathfrak{B}[s]\|, \|\Pi\| \Rightarrow \neg\neg C_1}}{\dfrac{\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|, \|\Gamma\|, \|\Pi\| \Rightarrow \neg\neg C_1}{\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|, \|\Gamma\|, \|\Pi\|, \neg C_1 \Rightarrow \bot}}}{\dfrac{\neg\neg(\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|), \|\Gamma\|, \|\Pi\|, \neg C_1 \Rightarrow \bot}{\dfrac{\neg\neg(\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|), \|\Gamma\|, \|\Pi\| \Rightarrow \neg\neg C_1}{\dfrac{\|\lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y], \|\Gamma\|, \|B\| \Rightarrow \|C\|}{\|\lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y], \Gamma, \Pi \Rightarrow C\|}}}} \text{ 1.3iii}$$

Re 1.8v. This is '$\sqsubseteq$-right'.

$$\dfrac{\dfrac{\dfrac{\dfrac{\|\Gamma, \mathfrak{A}[a] \Rightarrow \mathfrak{B}[a]\|}{\|\Gamma\|, \|\mathfrak{A}[a]\| \Rightarrow \|\mathfrak{B}[a]\|}}{\|\Gamma\| \Rightarrow \lambda x \, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y \, \|\mathfrak{B}[y]\|}}{\|\Gamma\| \Rightarrow \neg\neg(\lambda x \, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y \, \|\mathfrak{B}[y]\|)}}{\|\Gamma \Rightarrow \lambda x \, \mathfrak{A}[x] \sqsubseteq \lambda y \, \mathfrak{B}[y]\|}$$

QED

2. Addition 130d. Application of the fixed point property: a numeralwise representation of the recursive functions in $\mathbf{L^iD_\lambda}$[2]

The possibility of obtaining a definition of the natural numbers in $\mathbf{L^iD}_\lambda$ that would provide induction in a "second order style" as, *e.g.*, in section 41f in [**15**], is out of question for simple ordinal reasons: the consistency of $\mathbf{L^iD}_\lambda$ is already provable by means of a simple induction. As a result, the possibility of defining recursive functions in a "Dedekind style" is not open.[3]

There is, however, the possibility of numeralwise representing all recursive functions. This possibility is essentially based on two features of contraction free logic with unrestricted abstraction, *viz.*,

— the (direct) fixed point property, and
— the contractibility of $\equiv$-wffs.

The (direct) fixed point property provides for terms that numeralwise represent recursive functions somewhat like the recursion theorem provides for partial recursive functions.[4] What is specific about this numeralwise representation of recursive function is the role of identity; *i.e.*, what is

[2] This addition was sparked by [**18**] and [**19**]. Cf. also [**6**]. An actual proof of the numeralwise representability of the recursive functions does not seem to be available in print. [**18**] is not published and [**19**] only states the result with reference to [**18**].

[3] It is possible, of course, to provide definitions in that style, but due to the deductive weakness of $\mathbf{L^iD}_\lambda$ their characteristic properties cannot be proved in $\mathbf{L^iD}_\lambda$. As emphasized in [**19**], p. 10 (albeit with regard to a slightly different system), "such a theory is descriptively rich" but "proof theoretically very weak (as its consistency is established by the induction up to ω)."

[4] There is a significant difference, though: the recursion theorem is compatible with classical logic, but not so the (direct) fixed point theorem.

being considered are numerals, not anything that equals it.[5] In this way some valuable classical features are rescued for our non-classical situation like the very contractibility of $\equiv$-wffs.

This approach works well for all functions defined by n-recursion. It is the sort of closure operation constituted by minimization that needs special attention. What is required is a form of trichotomy in order to prove that minimization can be numeralwise represented.

The proximity of the proof presented here to the one in [**3**], pp. 192–199, or [**2**], pp. 166–171, for the case of Robinson's arithmetic will be obvious. The main point is that the smaller relation and with it the representation of the least number operater is based on a term $\mathbf{B}^*$ which is introduced as a fixed point. It acts like a strengthened kind of B-axiom[6] in that it allows to prove a form of trichotomy. As in the case of Robinson's arithmetic heavy weight lies on the use of meta-theoretical induction. That's where results are only established for numerals.

I begin with an adaptation of the notion of *numeralwise representation* to the situation of $\mathbf{L^iD}_\lambda$.

DEFINITIONS 2.1. (1) A k-place total function $\mathfrak{f}$ is said to be numeralwise represented by f in $\mathbf{L^iD}_\lambda$, if the following holds:

$$\text{if } \mathfrak{f}(\vec{\mathfrak{n}}) = \mathfrak{m}, \text{ then } \begin{cases} \vdash_{\mathbf{L^iD}_\lambda} \langle \vec{n}, m \rangle \in f \\ \vdash_{\mathbf{L^iD}_\lambda} \bigwedge x (\langle \vec{n}, x \rangle \in f \to x \equiv m) \end{cases}$$

for all k-tuples $\vec{\mathfrak{n}}$ of natural numbers and natural numbers $\mathfrak{m}$.
(2) A function $\mathfrak{f}$ is said to be *numeralwise representable in* $\mathbf{L^iD}_\lambda$, if there is a term t which numeralwise represents $\mathfrak{f}$ in $\mathbf{L^iD}_\lambda$.

Next come the exclusive successor notion and some of its properties which will be needed later.

DEFINITION 2.2. $s^{\imath} :\equiv \lambda x (x \in s \diamond x \equiv s)$.

---

[5] This means, in particular, that functions cannot be employed to apply to arguments; *i.e.*, , instead of $f[\![x]\!] = y$ one only has something like $\langle y, x \rangle \in f$.

[6] Cf. in definition 128.36 on p. 1764 of [**15**].

PROPOSITION 2.3. *Sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.3i) $\Rightarrow s \in s^{\prime}$

(2.3ii) $s^{\prime} \equiv 0 \Rightarrow$

(2.3iii) $s^{\prime} \equiv t^{\prime} \Rightarrow s \in t^{\prime}$

(2.3iv) $s \in n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n$

(2.3v) $s^{\prime} \equiv n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n$

(2.3vi) $s \equiv n^{\prime}, s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \Rightarrow$

(2.3vii) $s^{\prime} \equiv n^{\prime} \Rightarrow s \equiv n$

Proof. *Re* 2.3i and 2.3ii. As for their inclusive counterparts, cf. 128.29i and 128.29ii in [**15**], p. 1759.
Re 2.3iii.

$$\dfrac{\dfrac{\Rightarrow s \in s^{\prime} \qquad s \in t^{\prime} \Rightarrow s \in t^{\prime}}{s \in s^{\prime} \rightarrow s \in t^{\prime} \Rightarrow s \in t^{\prime}}}{s^{\prime} \equiv t^{\prime} \Rightarrow s \in t^{\prime}}$$

Re 2.3iv. Employ a meta-theoretical induction on n.
n = 0:

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{s \in 0 \Rightarrow \qquad s \equiv 0 \Rightarrow s^{\prime} \equiv 0^{\prime}}{s \in 0^{\prime} \Rightarrow s \equiv 0}}{s \in 0^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime}}}{} \\ \hline\hline s \in 0^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n \end{array}$$

n = m$^{\prime}$:

$$\dfrac{\dfrac{s \in m^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv m \qquad s \equiv m^{\prime} \Rightarrow s \equiv m^{\prime}}{s \in m^{\prime} \diamond s \equiv m^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv m \diamond s \equiv m^{\prime}}}{s \in m^{\prime\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv m \diamond s \equiv m^{\prime}}$$

Re 2.3v. Employ a cut on 2.3iii and 2.3iv:

$$\dfrac{s^{\prime} \equiv n^{\prime} \Rightarrow s \in n^{\prime} \qquad s \in n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n}{s^{\prime} \equiv n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n} \clubsuit$$

Re 2.3vi. Employ a meta-theoretical induction on n. For n = 0, the situation is immediately clear from 126.45i in [**15**]:

As regards $\mathrm{n} = \mathrm{m}^{\prime}$:

$$\dfrac{\dfrac{\dfrac{m^{\prime\prime} \equiv 0^{\prime} \Rightarrow}{s^{\prime} \equiv m^{\prime\prime}, s^{\prime} \equiv 0^{\prime} \Rightarrow} \quad \dots \quad \dfrac{m^{\prime\prime} \equiv m^{\prime} \Rightarrow}{s^{\prime} \equiv m^{\prime\prime}, s^{\prime} \equiv m^{\prime} \Rightarrow}}{\text{n } \diamond\text{-introductions left}}}{s^{\prime} \equiv m^{\prime\prime}, s^{\prime} \equiv 0 \diamond \cdots \diamond s^{\prime} \equiv m^{\prime} \Rightarrow}$$

Re 2.3vii. This is now an immediate consequence of 2.3v and 2.3vi:

$$\dfrac{s^{\prime} \equiv n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n \qquad s^{\prime} \equiv n^{\prime}, s^{\prime} \equiv 0 \diamond \cdots \diamond s^{\prime} \equiv n \Rightarrow}{s^{\prime} \equiv n^{\prime} \Rightarrow s \equiv n}$$

QED

In view of result 10.6 in [**15**], p. 77, it is sufficient to consider:

1. basic functions **Z**, **S**, **I**, and the characteristic function of equality
2. composition
3. addition and multiplication
4. minimization

I begin with a numeralwise representation of the functions listed under 1 and 2.

DEFINITIONS 2.4. (1) *zero* $:\equiv \lambda xy(y \equiv 0)$.
(2) *suc* $:\equiv \lambda xy(y \equiv x^{\prime})$.
(3) $id^{\mathrm{m}}_{\mathrm{n}} :\equiv \lambda \vec{x} y(y \equiv x_{\mathrm{n}})$.
(4) $char_{=} :\equiv \lambda xyz((x \equiv y \,\square\, z \equiv 0) \vee (x \not\equiv y \,\square\, z \equiv 1))$.
(4) $comp[h, \vec{g}] :\equiv \lambda \vec{x} \vec{y} z(\langle \vec{x}, y_1 \rangle \in g_1 \,\square\, \dots \,\square\, \langle \vec{x}, y_{\mathrm{n}} \rangle \in g_{\mathrm{n}} \,\square\, \langle \vec{y}, z \rangle \in h)$.

REMARK 2.5. In view of the definition of $\lambda xy\, \mathfrak{F}[x, y]$, the definition of *zero*, *e.g.*, amounts to $\lambda z \bigvee x \bigvee y(z \equiv \langle x, y \rangle \,\square\, y \equiv 0)$.[7]

PROPOSITION 2.6.

(2.6i) *zero numeralwise represents the zero function* **Z**

(2.6ii) *suc numeralwise represents the successor function* **S**

(2.6iii) *id numeralwise represents* **I**

[7] The axiom employed in [**18**] amounts to $\lambda z \bigvee x(z \equiv \langle x, 0 \rangle)$ in my symbolism.

(2.6iv) *$char_=$ numeralwise represents the characteristic function of equality* χ_{eq}

(2.6v) *$comp[h, \vec{g}]$ numeralwise represents the composition of functions* $\mathrm{Cn}[h, g_1, \ldots, g_{\mathrm{m}}]$

Proof. Completely straightforward, but to see the point of the notion of identity in the definitions, I just indicate how to treat the case of *zero*: What has to be shown is

$$\vdash_{\mathbf{L^iD_\lambda}} \langle n, 0\rangle \in \lambda xy(y \equiv 0), \text{ and}$$
$$\vdash_{\mathbf{L^iD_\lambda}} \bigwedge x(\langle n, x\rangle \in zero \rightarrow x \equiv 0)\,.$$

The first one reduces to $0 \equiv 0$ and the second one to $a \equiv 0 \Rightarrow a \equiv 0$. QED

PROPOSITION 2.7. *There are terms add and mult satisfying*

$$(2.7\mathrm{i})\quad \mathbf{L^iD_\lambda} \vdash\ add = \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv x_1)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{\prime} \,\square\, x_3 \equiv z^{\prime} \,\square\, \langle\langle x_1, y\rangle, z\rangle \in add))$$

$$(2.7\mathrm{ii})\quad \mathbf{L^iD_\lambda} \vdash\ mult = \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv 0)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{\prime} \,\square\, \langle\langle z, x_1\rangle, x_3\rangle \in add \,\square\, \langle\langle x_1, y\rangle, z\rangle \in mult))$$

Proof. This is again an immediate consequence of the fixed point property. QED

The following convention is introduced for the convenience of formulating results regarding *add* and *mult*.

CONVENTION 2.8.

$$(1)\quad \mathcal{ADD} :\equiv \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv x_1)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{\prime} \,\square\, x_3 \equiv z^{\prime} \,\square\, \langle\langle x_1, y\rangle, z\rangle \in add))$$

$$(2)\quad \mathcal{MULT} :\equiv \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv 0)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{\prime} \,\square\, \langle\langle z, x_1\rangle, x_3\rangle \in add \,\square\, \langle\langle x_1, y\rangle, z\rangle \in mult))$$

COROLLARY 2.9. *Inferences according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-derivable*

(2.9i) $$\frac{s \in \mathcal{ADD}, \Gamma \Rightarrow C}{s \in add, \Gamma \Rightarrow C}$$

(2.9ii) $$\frac{\Gamma \Rightarrow s \in \mathcal{ADD}}{\Gamma \Rightarrow s \in add}$$

(2.9iii) $$\frac{s \in \mathcal{MULT}, \Gamma \Rightarrow C}{s \in mult, \Gamma \Rightarrow C}$$

(2.9iv) $$\frac{\Gamma \Rightarrow s \in \mathcal{MULT}}{\Gamma \Rightarrow s \in mult}$$

PROPOSITION 2.10. *Sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.10i) $\Rightarrow \langle\langle s, 0\rangle, s\rangle \in add$

(2.10ii) $\langle\langle s, t\rangle, r\rangle \in add \Rightarrow \langle\langle s, t^{\prime}\rangle, r^{\prime}\rangle \in add$

(2.10iii) $\langle\langle s, 0\rangle, t\rangle \in add \Rightarrow t \equiv s$

(2.10iv) $\bigwedge x\,(\langle\langle s, n\rangle, x\rangle \in add \rightarrow x \equiv p), \langle\langle s, n^{\prime}\rangle, t\rangle \in add \Rightarrow t \equiv p^{\prime}$

Proof. *Re* 2.10i.

$$\cfrac{\cfrac{\cfrac{\Rightarrow 0 \equiv 0 \qquad \Rightarrow s \equiv s}{\Rightarrow 0 \equiv 0 \Box s \equiv s}}{\Rightarrow (0 \equiv 0 \Box s \equiv s) \Diamond \bigvee y \bigvee z\,(0 \equiv y^{\prime} \Box s \equiv z^{\prime} \Box \langle\langle s, y\rangle, z\rangle \in add)}}{\Rightarrow \langle\langle s, 0\rangle, s\rangle \in add}$$

Re 2.10ii. In view of 2.19i below, this is left to the reader.

Re 2.10iii.

$$\dfrac{\dfrac{\dfrac{t \equiv s \Rightarrow t \equiv s}{0 \equiv 0, t \equiv s \Rightarrow t \equiv s}}{0 \equiv 0 \,\Box\, t \equiv s \Rightarrow t \equiv s} \qquad \dfrac{\dfrac{\dfrac{0 \equiv b' \Rightarrow}{0 \equiv b', t \equiv c', \langle\langle s,b\rangle, c\rangle \in add) \Rightarrow}}{0 \equiv b' \,\Box\, t \equiv c' \,\Box\, \langle\langle s,b\rangle, c\rangle \in add) \Rightarrow}}{\bigvee y \bigvee z\,(0 \equiv y' \,\Box\, t \equiv z' \,\Box\, \langle\langle s,y\rangle, z\rangle \in add) \Rightarrow}}{\dfrac{(0 \equiv 0 \,\Box\, t \equiv s) \diamond \bigvee y \bigvee z\,(0 \equiv y' \,\Box\, t \equiv z' \,\Box\, \langle\langle s,y\rangle, z\rangle \in add) \Rightarrow t \equiv s}{\langle\langle s,0\rangle, t\rangle \in add \Rightarrow t \equiv s}\;2.9\text{ii}}$$

Re 2.10iv. Let $\mathcal{A}$ stand for $\bigwedge x\,(\langle\langle s,n\rangle, x\rangle \in add \rightarrow x \equiv p)$ and $\mathcal{C}$ for $\bigvee y \bigvee z\,(n' \equiv y' \,\Box\, t \equiv z' \,\Box\, \langle\langle s,y\rangle, z\rangle \in add)$:

$$\dfrac{\dfrac{\Rightarrow \neg(n' \equiv 0 \,\Box\, t \equiv s) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle\langle s,n\rangle, c\rangle \in add \Rightarrow \langle\langle s,n\rangle, c\rangle \in add \qquad \dfrac{c \equiv p \Rightarrow c' \equiv p'}{c \equiv p, t \equiv c' \Rightarrow t \equiv p'}}{\langle\langle s,n\rangle, c\rangle \in add \rightarrow c \equiv p, t \equiv c', \langle\langle s,n\rangle, c\rangle \in add \Rightarrow t \equiv p'}}{\mathcal{A}, t \equiv c', \langle\langle s,n\rangle, c\rangle \in add \Rightarrow t \equiv p'}}{\mathcal{A}, n \equiv b, t \equiv c', \langle\langle s,b\rangle, c\rangle \in add \Rightarrow t \equiv p'}\;2.3\text{vii}}{\mathcal{A}, n' \equiv b', t \equiv c', \langle\langle s,b\rangle, c\rangle \in add \Rightarrow t \equiv p'}}{\mathcal{A}, n' \equiv b' \,\Box\, t \equiv c' \,\Box\, \langle\langle s,b\rangle, c\rangle \in add \Rightarrow t \equiv p'}}{\bigwedge x\,(\langle\langle s,n\rangle, x\rangle \in add \rightarrow x \equiv p), \mathcal{C} \Rightarrow t \equiv p'}}{\bigwedge x\,(\langle\langle s,n\rangle, x\rangle \in add \rightarrow x \equiv p), (n' \equiv 0 \,\Box\, t \equiv s) \diamond \mathcal{C} \Rightarrow t \equiv p'}}{\bigwedge x\,(\langle\langle s,n\rangle, x\rangle \in add \rightarrow x \equiv p), \langle\langle s,n'\rangle, t\rangle \in add \Rightarrow t \equiv p'}\;2.9\text{ii}$$

QED

PROPOSITION 2.11. *Sequents according to the following schemata are* $\mathbf{L^iD}_\lambda$*-deducible.*

(2.11i) $\Rightarrow \langle\langle s,0\rangle, 0\rangle \in mult$

(2.11ii) $\langle\langle s,t\rangle, s_1\rangle \in mult, \langle\langle s_1,s\rangle, r\rangle \in add \Rightarrow \langle\langle s,t'\rangle, r\rangle \in mult$

(2.11iii) $\langle\langle s,0\rangle, t\rangle \in mult \Rightarrow t \equiv 0$

(2.11iv) $\bigwedge x\,(\langle\langle s,n\rangle, x\rangle \in mult \rightarrow x \equiv r_1), \langle\langle s,n'\rangle, t\rangle \in mult,$
$\bigwedge x\,(\langle\langle r_1,s\rangle, x\rangle \in add \rightarrow x \equiv r_2) \Rightarrow t \equiv r_2$

Proof. Essentially as for 2.10; I shall only treat the second as an example. *Re* 2.11ii. To save space, let $\mathcal{C}$ be short for $(t^{\prime} \equiv 0 \Box r \equiv 0)$ and $\mathcal{A}$ for $\langle\langle s, t\rangle, s_1\rangle \in mult$:

$$\mathcal{A} \Rightarrow t^{\prime} \equiv t^{\prime} \Box \langle\langle s, t\rangle, s_1\rangle \in mult \qquad \langle\langle s_1, s\rangle, r\rangle \in add \Rightarrow \langle\langle s_1, s\rangle, r\rangle \in add$$

$$\langle\langle s, t\rangle, s_1\rangle \in mult, \langle\langle s_1, s\rangle, r\rangle \in add \Rightarrow t^{\prime} \equiv t^{\prime} \Box \langle\langle s_1, s\rangle, r\rangle \in add \Box \langle\langle s, t\rangle, s_1\rangle \in mult$$

$$\mathcal{A}, \langle\langle s_1, s\rangle, r\rangle \in add \Rightarrow \bigvee y \bigvee z\, (t^{\prime} \equiv y^{\prime} \Box \langle\langle z, s\rangle, r\rangle \in add \Box \langle\langle x_1, y\rangle, z\rangle \in mult)$$

$$\mathcal{A}, \langle\langle s_1, s\rangle, r\rangle \in add \Rightarrow \mathcal{C} \diamond \bigvee y \bigvee z\, (t^{\prime} \equiv y^{\prime} \Box \langle\langle z, s\rangle, r\rangle \in add \Box \langle\langle x_1, y\rangle, z\rangle \in mult)$$

$$\langle\langle s, t\rangle, s_1\rangle \in mult, \langle\langle s_1, s\rangle, r\rangle \in add \Rightarrow \langle\langle s, t^{\prime}\rangle, r\rangle \in mult$$

Re 2.11iv. To save space, let $\mathcal{A}$ stand for $\bigwedge x\, (\langle\langle s, n\rangle, x\rangle \in mult \rightarrow x \equiv r_1)$ and $\mathcal{C}$ for $\bigvee y \bigvee z\, (n^{\prime} \equiv y^{\prime} \Box \langle\langle z, s\rangle, t\rangle \in add \Box \langle\langle s, y\rangle, z\rangle \in mult)$ and $\mathcal{F}$ for $\bigwedge x\, (\langle\langle r_1, s\rangle, x\rangle \in add \rightarrow x \equiv r_2)$:

$$\langle\langle r_1, s\rangle, t\rangle \in add \Rightarrow \langle\langle r_1, s\rangle, t\rangle \in add \qquad t \equiv r_2 \Rightarrow t \equiv r_2$$

$$\langle\langle r_1, s\rangle, t\rangle \in add \rightarrow t \equiv r_2, \langle\langle r_1, s\rangle, t\rangle \in add \Rightarrow t \equiv r_2$$

$$\mathcal{F}, \langle\langle r_1, s\rangle, t\rangle \in add \Rightarrow t \equiv r_2$$

$$\langle\langle s, n\rangle, c\rangle \in mult \Rightarrow \langle\langle s, n\rangle, c\rangle \in mult \qquad c \equiv r_1, \mathcal{F}, \langle\langle c, s\rangle, t\rangle \in add \Rightarrow t \equiv r_2$$

$$\langle\langle s, n\rangle, c\rangle \in mult \rightarrow c \equiv r_1, \langle\langle c, s\rangle, t\rangle \in add, \langle\langle s, n\rangle, c\rangle \in mult, \mathcal{F} \Rightarrow t \equiv r_2$$

$$\mathcal{A}, \langle\langle c, s\rangle, t\rangle \in add, \langle\langle s, n\rangle, c\rangle \in mult, \mathcal{F} \Rightarrow t \equiv r_2$$

$$\mathcal{A}, n \equiv b, \langle\langle c, s\rangle, t\rangle \in add, \langle\langle s, b\rangle, c\rangle \in mult, \mathcal{F} \Rightarrow t \equiv r_2$$

2.3vii

$$\mathcal{A}, n^{\prime} \equiv b^{\prime}, \langle\langle c, s\rangle, t\rangle \in add, \langle\langle s, b\rangle, c\rangle \in mult, \mathcal{F} \Rightarrow t \equiv r_2$$

$$\mathcal{A}, n^{\prime} \equiv b^{\prime} \Box \langle\langle c, s\rangle, t\rangle \in add \Box \langle\langle s, b\rangle, c\rangle \in mult, \mathcal{F} \Rightarrow t \equiv r_2$$

$$\Rightarrow \neg(n^{\prime} \equiv 0 \Box t \equiv 0) \qquad \mathcal{A}, \mathcal{C}, \mathcal{F} \Rightarrow t \equiv r_2$$

$$\mathcal{A}, \neg(n^{\prime} \equiv 0 \Box n \equiv 0) \diamond \mathcal{C}, \mathcal{F} \Rightarrow t \equiv r_2$$

2.9iv

$$\mathcal{A}, \langle\langle s, n^{\prime}\rangle, t\rangle \in mult, \mathcal{F} \Rightarrow t \equiv r_2$$

QED

PROPOSITION 2.12.

(2.12i) *add numeralwise represents the function* $+$

(2.12ii) *mult numeralwise represents the function* $\cdot$

Proof. *Re* 2.12i. What has to be shown is that if m + n = p, then

$$\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, n\rangle, p\rangle \in \mathit{add}, \text{ and}$$
$$\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, n\rangle, x\rangle \in \mathit{add} \to x \equiv p)\,.$$

In both cases, employ a meta-theoretical induction on n.
As regards the first one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, 0\rangle, m\rangle \in \mathit{add}$. This is 2.10i.
n = k′. What has to be shown is that if p is the numerical value of m + k, then $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, k^{\prime}\rangle, p^{\prime}\rangle \in \mathit{add}$. By the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, k\rangle, p\rangle \in \mathit{add}$. This yields the claim by a cut with 2.10ii.
As regards the second one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, 0\rangle, x\rangle \in \mathit{add} \to x \equiv m)$. This is easily obtained from 2.10iii.
n = k′. What has to be shown is that if p is the numerical value of m + k, then $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, k^{\prime}\rangle, x\rangle \in \mathit{add} \to x \equiv p^{\prime})$. By the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, k\rangle, x\rangle \in \mathit{add} \to x \equiv p)$. By a cut with 2.10ii this yields $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, k^{\prime}\rangle, c\rangle \in \mathit{add} \Rightarrow c \equiv p^{\prime}$ which yields the claim by →- and $\bigwedge$-introduction.
Re 2.12ii. What has to be shown is that if m · n = p, then

$$\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, n\rangle, p\rangle \in \mathit{mult}, \text{ and}$$
$$\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, n\rangle, x\rangle \in \mathit{mult} \to x \equiv p)\,.$$

Again, employ meta-theoretical inductions on n.
As regards the first one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, 0\rangle, 0\rangle \in \mathit{mult}$. This is 2.11i.
n = k′. What has to be shown is that if p is the numerical value of m · k and q is the numerical value of p + m, then $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, k^{\prime}\rangle, q\rangle \in \mathit{mult}$. By the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, k\rangle, p\rangle \in \mathit{mult}$ and by 2.12i, $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle p, m\rangle, q\rangle \in \mathit{add}$. Two cuts with 2.11ii yield the claim.
As regards the second one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, 0\rangle, x\rangle \in \mathit{mult} \to x \equiv 0)$. This is easily obtained from 2.11iii.
n = k′. Let p be the numerical value of m · k and q that of p + m. Then, by the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, k\rangle, x\rangle \in \mathit{mult} \to x \equiv p)$ and by 2.12i, $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle p, m\rangle, x\rangle \in \mathit{add} \to x \equiv q)$. Two cuts with 2.11iv yield $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m, k^{\prime}\rangle, t\rangle \in \mathit{mult} \Rightarrow t \equiv q$. Applying a →- and a $\bigwedge$-introduction then yields $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m, k^{\prime}\rangle, x\rangle \in \mathit{mult} \to x \equiv q)$. QED

REMARK 2.13. Of course, *all* total functions definable by 1-recursion can be numeralwise represented in that way. If the function $\mathfrak{f}$ is defined by primitive recursion from the functions $\mathfrak{g}$ and $\mathfrak{h}$, and $\mathfrak{g}$ and $\mathfrak{h}$ are represented in $\mathbf{L^iD}_\lambda$ by g and h, respectively, then $\mathfrak{f}$ is represented by the term f satisfying the following fixed point property in $\mathbf{L^iD}_\lambda$:

$$\begin{aligned} f = \lambda x_1 x_2 x_3\, ((x_2 \equiv 0 \,\Box\, \langle x_1, x_3\rangle \in g) \diamond \\ \bigvee y \bigvee z\, (x_2 \equiv y' \,\Box\, \langle\langle x_1, y\rangle, z\rangle \in f \,\Box\, \langle\langle\langle x_1, y\rangle, z\rangle, x_3\rangle \in h))\,. \end{aligned}$$

As a matter of fact, n-recursion can be represented in that way too. As an example, consider the so-called *Ackermann function*. Employ the following fixed point *ak* for a numeralwise representation of the Ackermann function:

$$\begin{aligned} ak = \lambda x_1 x_2 x_3\, ((x_1 \equiv 0 \,\Box\, x_3 \equiv x_2{}') \diamond \\ \bigvee y\, (x_1 \equiv y' \,\Box\, x_2 \equiv 0 \,\Box\, \langle\langle y, 0'\rangle, x_3\rangle \in ak) \diamond \\ \bigvee y_1 \bigvee y_2 \bigvee z\, (x_1 \equiv y_1{}' \,\Box\, x_2 \equiv y_2{}' \,\Box\, \langle\langle x_1, y_2\rangle, z\rangle \in ak \,\Box\, \langle\langle x_1, z\rangle, x_3\rangle \in ak)\,. \end{aligned}$$

In other words, all stages of recursion can be numeralwise represented in a straightforward manner. This may provoke the question as to what the least number operator actually adds to the notion of recursion.

The following schemata of inference will come handy in the further presentation. They are instances of what I called an "exclusion principle" in remarks 116.6 and 119.1 in [**15**], for example.

PROPOSITION 2.14. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda$*-derivable.*

(2.14i)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[s, 0, s] \qquad \Gamma, \langle\langle s, a\rangle, b\rangle \in add \Rightarrow \mathfrak{F}[s, a', b']}{\Gamma, \langle\langle s, t\rangle, r\rangle \in add \Rightarrow \mathfrak{F}[s, t, r]}$$

(2.14ii)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[s, 0, s] \qquad \Gamma, \langle\langle b, s\rangle, r\rangle \in add, \langle\langle s, a\rangle, b\rangle \in mult \Rightarrow \mathfrak{F}[s, a', r]}{\Gamma, \langle\langle s, t\rangle, r\rangle \in mult \Rightarrow \mathfrak{F}[s, t, r]}$$

Proof. *Re* 2.14i.

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \mathfrak{F}[s,0,s]}{\Gamma, t\equiv 0, r\equiv s \Rightarrow \mathfrak{F}[s,t,r]}\ \text{(double line)}}{\Gamma, t\equiv 0 \Box r\equiv s \Rightarrow \mathfrak{F}[s,t,r]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \langle\!\langle s,a\rangle, b\rangle \in \mathit{add} \Rightarrow \mathfrak{F}[s,a',b']}{\Gamma, t\equiv a', r\equiv b', \langle\!\langle s,a\rangle, b\rangle \in \mathit{add} \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, t\equiv a' \Box r\equiv b' \Box \langle\!\langle s,a\rangle, b\rangle \in \mathit{add} \Rightarrow \mathfrak{F}[s,t,r]}\ \text{(double line)}}{\Gamma, \bigvee y \bigvee z\,(t\equiv y' \Box r\equiv z' \Box \langle\!\langle s,y\rangle, z\rangle \in \mathit{add}) \Rightarrow \mathfrak{F}[s,t,r]}\ \text{(double line)}}{\dfrac{\Gamma, (t\equiv 0 \Box r\equiv s) \diamond \bigvee y \bigvee z\,(t\equiv y' \Box r\equiv z' \Box \langle\!\langle s,y\rangle, z\rangle \in \mathit{add}) \Rightarrow \mathfrak{F}[s,t,r]}{\Gamma, \langle\!\langle s,t\rangle, r\rangle \in \mathit{add} \Rightarrow \mathfrak{F}[s,t,r]}}$$

Re 2.14ii. Let $\mathfrak{A} :\equiv (t \equiv {*_1}' \Box \langle\!\langle *_2, s\rangle, r\rangle \in \mathit{add} \Box \langle\!\langle s, *_1\rangle, *_2\rangle \in \mathit{mult})$:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \mathfrak{F}[s,0,0]}{\Gamma, t\equiv 0, r\equiv 0 \Rightarrow \mathfrak{F}[s,t,r]}\ \text{(double line)}}{\Gamma, t\equiv 0 \Box r\equiv 0 \Rightarrow \mathfrak{F}[s,t,r]} \qquad \dfrac{\dfrac{\dfrac{\Gamma, \langle\!\langle b,s\rangle, r\rangle \in \mathit{add}, \langle\!\langle s,a\rangle, b\rangle \in \mathit{mult} \Rightarrow \mathfrak{F}[s,a',r]}{\Gamma, t\equiv a', \langle\!\langle b,s\rangle, r\rangle \in \mathit{add}, \langle\!\langle s,a\rangle, b\rangle \in \mathit{mult} \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, \mathfrak{A}[a,b] \Rightarrow \mathfrak{F}[s,t,r]}\ \text{(double line)}}{\Gamma, \bigvee y \bigvee z\, \mathfrak{A}[y,z] \Rightarrow \mathfrak{F}[s,t,r]}\ \text{(double line)}}{\dfrac{\Gamma, (t\equiv 0 \Box r\equiv 0) \diamond \bigvee y \bigvee z\, \mathfrak{A}[y,z] \Rightarrow \mathfrak{F}[s,t,r]}{\Gamma, \langle\!\langle s,t\rangle, r\rangle \in \mathit{mult} \Rightarrow \mathfrak{F}[s,t,r]}}$$

QED

PROPOSITION 2.15. *Sequents according to the following schemata are* $\mathbf{L^iD}_\lambda$*-deducible.*

(2.15i) $\langle\!\langle s', n\rangle, t\rangle \in \mathit{add} \Rightarrow \langle\!\langle s, n'\rangle, t\rangle \in \mathit{add}$

(2.15ii) $\langle\!\langle c', a\rangle, n'\rangle \in \mathit{add} \Rightarrow a \equiv 0 \diamond \ldots \diamond a \equiv n$

Proof. *Re* 2.15i. Employ an induction on n. As regards the induction basis, employ 2.10iii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle\!\langle s',0\rangle, t\rangle \in \mathit{add} \Rightarrow t \equiv s'}{\langle\!\langle s',0\rangle, t\rangle \in \mathit{add} \Rightarrow 0' \equiv 0' \Box t \equiv s' \Box \langle\!\langle s,0\rangle, s\rangle \in \mathit{add}}\ \text{(double line)}}{\langle\!\langle s',0\rangle, t\rangle \in \mathit{add} \Rightarrow \bigvee y \bigvee z\,(0' \equiv y' \Box t \equiv z' \Box \langle\!\langle s,y\rangle, z\rangle \in \mathit{add})}}{\langle\!\langle s',0\rangle, t\rangle \in \mathit{add} \Rightarrow (0' \equiv 0 \Box t \equiv s) \diamond \bigvee y \bigvee z\,(0' \equiv y' \Box t \equiv z' \Box \langle\!\langle s,y\rangle, z\rangle \in \mathit{add})}}{\langle\!\langle s',0\rangle, t\rangle \in \mathit{add} \Rightarrow \langle\!\langle s,0'\rangle, t\rangle \in \mathit{add}}$$

As regards the induction step, firstly, employ 2.3ii:

$$\begin{array}{c}
0 \equiv n' \Rightarrow \\
\hline
0 \equiv n', t \equiv s' \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\
\hline\hline
0 \equiv n' \square t \equiv s' \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add
\end{array}$$

Secondly, employ the induction hypothesis. In the proof figure to follow, let $\mathcal{C}$ stand for $(n'' \equiv 0 \square t \equiv s)$:

$$\begin{array}{c}
\begin{array}{c} t \equiv c' \Rightarrow t \equiv c' \\ \hline t \equiv c' \square \langle\langle s', n\rangle, c\rangle \in add \Rightarrow t \equiv c' \end{array}
\qquad
\begin{array}{c} \langle\langle s', n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n'\rangle, c\rangle \in add \\ \hline t \equiv c' \square \langle\langle s', n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n'\rangle, c\rangle \in add \end{array} \\
\hline
t \equiv c' \square \langle\langle s', n\rangle, c\rangle \in add \Rightarrow t \equiv c' \square \langle\langle s, n'\rangle, c\rangle \in add \\
\hline
t \equiv c' \square \langle\langle s', b\rangle, c\rangle \in add \Rightarrow n'' \equiv n'' \square t \equiv c' \square \langle\langle s, n'\rangle, c\rangle \in add \\
\hline\hline
t \equiv c' \square \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \bigvee y \bigvee z\,(n'' \equiv y' \square t \equiv z' \square \langle\langle s, y\rangle, z\rangle \in add) \\
\hline
t \equiv c' \square \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \mathcal{C} \diamond \bigvee y \bigvee z\,(n'' \equiv y' \square t \equiv z' \square \langle\langle s, y\rangle, z\rangle \in add) \\
\hline
t \equiv c' \square \langle\langle s', n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\
\hline
n \equiv b, t \equiv c' \square \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\
\hline
n' \equiv b', t \equiv c' \square \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\
\hline
n' \equiv b' \square t \equiv c' \square \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\
\hline\hline
\bigvee y \bigvee z\,(n' \equiv y' \square t \equiv z' \square \langle\langle s', y\rangle, z\rangle \in add) \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add
\end{array}$$

Together:

$$\begin{array}{c}
(0 \equiv n' \square t \equiv s') \diamond \bigvee y \bigvee z\,(n' \equiv y' \square t \equiv z' \square \langle\langle s', y\rangle, z\rangle \in add) \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\
\hline
\langle\langle s', n'\rangle, t\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add
\end{array}$$

Re 2.15ii. Employ an induction on n. I only consider the induction step. Let $\mathfrak{E}$ stand for $*_1 \equiv *_2 \diamond \cdots \diamond *_1 \equiv n'$:

$$
\dfrac{
  \dfrac{0 \equiv c^{\prime} \Rightarrow}{s \equiv 0 \,\square\, n^{\prime} \equiv c^{\prime} \Rightarrow s \equiv 0}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\langle\!\langle\langle\!\langle c^{\prime}, a\rangle\!\rangle, n^{\prime}\rangle\!\rangle \in add \Rightarrow a \equiv 0 \diamond \ldots \diamond a \equiv n}
                {\langle\!\langle\langle\!\langle c^{\prime}, a\rangle\!\rangle, n^{\prime}\rangle\!\rangle \in add \Rightarrow a^{\prime} \equiv 0^{\prime} \diamond \ldots \diamond a^{\prime} \equiv n^{\prime}}
        }{s \equiv a^{\prime}, n^{\prime} \equiv b, \langle\!\langle\langle\!\langle c^{\prime}, a\rangle\!\rangle, b\rangle\!\rangle \in add \Rightarrow \mathfrak{E}[s, 0^{\prime}]}
      }{s \equiv a^{\prime}, n^{\prime\prime} \equiv b^{\prime}, \langle\!\langle\langle\!\langle c^{\prime}, a\rangle\!\rangle, b\rangle\!\rangle \in add \Rightarrow \mathfrak{E}[s, 0^{\prime}]}\ 2.3\mathrm{vii}
    }{s \equiv a^{\prime} \,\square\, n^{\prime\prime} \equiv b^{\prime} \,\square\, \langle\!\langle\langle\!\langle c^{\prime}, a\rangle\!\rangle, b\rangle\!\rangle \in add \Rightarrow \mathfrak{E}[s, 0^{\prime}]}
  }{\bigvee y \bigvee z\, (s \equiv y^{\prime} \,\square\, n^{\prime\prime} \equiv z^{\prime} \,\square\, \langle\!\langle\langle\!\langle c^{\prime}, y\rangle\!\rangle, z\rangle\!\rangle \in add) \Rightarrow \mathfrak{E}[s, 0^{\prime}]}
}{
  \dfrac{(s \equiv 0 \,\square\, n^{\prime\prime} \equiv c^{\prime}) \diamond \bigvee y \bigvee z\, (s \equiv y^{\prime} \,\square\, n^{\prime\prime} \equiv z^{\prime} \,\square\, \langle\!\langle\langle\!\langle c^{\prime}, y\rangle\!\rangle, z\rangle\!\rangle \in add) \Rightarrow \mathfrak{E}[s, 0]}
        {\langle\!\langle\langle\!\langle c^{\prime}, s\rangle\!\rangle, n^{\prime\prime\prime}\rangle\!\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n^{\prime}}
}
$$

QED

The essential point for a representation of the least number operator is the availability of a smaller relation $<$ satisfying the following three conditions for all terms s and every numeral n:[8]

1. $\neg(s < 0)$
2. $s < n' \leftrightarrow s = 0 \vee \ldots \vee s = n$
3. $s < n \vee s = n \vee n < s$

i.e., a certain trichotomy of the natural numbers: two natural numbers are either equal or one of them is smaller than the other. In order to suit the present framework, the various notions involved have to be adapted. Equality will be replaced by identity, the inclusive successor will be replaced by the exclusive successor, $\vee$ will be replaced by $\diamond$. The task left is to find an appropriate notion of $<$. That's where a rudimentary[9] notion of natural number comes into play: m is smaller than n, if there is a natural number p such that m + p = n.

What has to be accommodated for is a certain self-reference in the definition of the natural numbers which is expressed in the simple statement: n is a *natural number*, if it is either 0 or the successor of a natural number. In other words, *natural number* is defined in terms of itself. This is what the fixed point of the next proposition aims at.

PROPOSITION 2.16. *There is a term* $\mathbf{B}^*$ *satisfying*

$$\mathbf{L^iD}_\lambda \vdash \mathbf{B}^* = \lambda x\, (x \equiv 0 \diamond \bigvee y\, (y \in \mathbf{B}^* \,\square\, x \equiv y^{\imath}))\,.$$

---

[8] Cf. [**13**], p. 40, proposition I.3.3. Note, however, that the $\wedge$ in the third condition listed there is obviously a typographical error that has to be replaced by $\vee$.

[9] 'Rudimentary', because full induction is not required.

Proof. This is an immediate consequence of the fixed point property as stated, *e.g.*, in [**14**], theorem 7.3, p. 382, or theorem 130.8 on p. 1779 of [**15**]. QED

COROLLARY 2.17. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda$*-derivable*

(2.17i)
$$\frac{s \in \lambda x(x \equiv 0 \diamond \bigvee y(y \in \mathbf{B}^* \,\square\, x \equiv y^{\prime})), \Gamma \Rightarrow C}{s \in \mathbf{B}^*, \Gamma \Rightarrow C}$$

(2.17ii)
$$\frac{\Gamma \Rightarrow s \in \lambda x(x \equiv 0 \diamond \bigvee y(y \in \mathbf{B}^* \,\square\, x \equiv y^{\prime}))}{\Gamma \Rightarrow s \in \mathbf{B}^*}$$

DEFINITION 2.18. $\bigvee^{\mathbf{B}^*} x\, \mathfrak{F}[x] :\equiv \bigvee x(x \in \mathbf{B}^* \,\square\, \mathfrak{F}[x])$.

I begin by listing the relevant properties of $\mathbf{B}^*$.

PROPOSITION 2.19. *Sequents according to the following schemata are* $\mathbf{L^iD}_\lambda$*-deducible.*

(2.19i) $\Rightarrow 0 \in \mathbf{B}^*$

(2.19ii) $s \in \mathbf{B}^* \Rightarrow s^{\prime} \in \mathbf{B}^*$

Proof. Re 2.19i. Employ 2.16:

$$\cfrac{\cfrac{\cfrac{\Rightarrow 0 \equiv 0}{\Rightarrow 0 \equiv 0 \diamond \bigvee^{\mathbf{B}^*} y\,(0 \equiv y^{\prime})}}{\Rightarrow 0 \in \lambda x\,(x \equiv 0 \diamond \bigvee^{\mathbf{B}^*} y\,(x \equiv y^{\prime}))}}{\Rightarrow 0 \in \mathbf{B}^*}\;2.17\text{ii}\,.$$

Re 2.19ii. Employ 2.16:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{s \in \mathbf{B}^* \Rightarrow s \in \mathbf{B}^* \qquad \Rightarrow s^{\prime} \equiv s^{\prime}}{s \in \mathbf{B}^* \Rightarrow s \in \mathbf{B}^* \,\square\, s^{\prime} \equiv s^{\prime}}}{s \in \mathbf{B}^* \Rightarrow \bigvee^{\mathbf{B}^*} y\,(s^{\prime} \equiv y^{\prime})}}{s \in \mathbf{B}^* \Rightarrow s^{\prime} \equiv 0 \diamond \bigvee^{\mathbf{B}^*} y\,(s^{\prime} \equiv y^{\prime})}}{s \in \mathbf{B}^* \Rightarrow s^{\prime} \in \lambda x\,(x \equiv 0 \diamond \bigvee^{\mathbf{B}^*} y\,(x \equiv y^{\prime}))}}{s \in \mathbf{B}^* \Rightarrow s^{\prime} \in \mathbf{B}^*}\;2.17\text{ii}\,.$$

QED

PROPOSITION 2.20. *Inferences according to the following schema are* $\mathbf{L^{i}D}_{\lambda}$*-derivable.*

$$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad \Gamma, a\in\mathbf{B}^* \Rightarrow \mathfrak{F}[a']}{\Gamma, s\in\mathbf{B}^* \Rightarrow \mathfrak{F}[s]}$$

Proof.

$$\cfrac{\cfrac{\cfrac{\Gamma \Rightarrow \mathfrak{F}[0]}{\Gamma, s\equiv 0 \Rightarrow \mathfrak{F}[s]} \qquad \cfrac{\cfrac{\cfrac{\Gamma, a\in\mathbf{B}^* \Rightarrow \mathfrak{F}[a']}{\Gamma, a\in\mathbf{B}^*, s\equiv a' \Rightarrow \mathfrak{F}[s]}}{\Gamma, a\in\mathbf{B}^* \sqcap s\equiv a' \Rightarrow \mathfrak{F}[s]}}{\Gamma, \bigvee^{\mathbf{B}^*}y\,(s\equiv y') \Rightarrow \mathfrak{F}[s]}}{\cfrac{\Gamma, s\equiv 0 \diamond \bigvee^{\mathbf{B}^*}y\,(s\equiv y') \Rightarrow \mathfrak{F}[s]}{\Gamma, s\in\lambda x(x\equiv 0 \diamond \bigvee^{\mathbf{B}^*}y\,(x\equiv y')) \Rightarrow \mathfrak{F}[s]}}}{\Gamma, s\in\mathbf{B}^* \Rightarrow \mathfrak{F}[s]}\ 2.17\mathrm{i}\,.$$

QED

For minimization a smaller-relation between numerals is required which is introduced next (essentially taken from [**18**], p. 8):

DEFINITION 2.21. $less :\equiv \lambda xy \bigvee^{\mathbf{B}^*}z\,(\langle\langle z', x\rangle, y\rangle \in add)\,.$

PROPOSITION 2.22. *If* m *and* n *are two natural numbers such that* m < n*, then* $\Rightarrow \langle m, n\rangle \in less$ *is* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

Proof. If m < n, then there is a natural number p such that $\mathrm{p}' + \mathrm{m} = \mathrm{n}$. By 2.19, $\Rightarrow p\in\mathbf{B}^*$ is $\mathbf{L^{i}D}_{\lambda}$-deducible and by the numeralwise representability of addition, $\Rightarrow \langle\langle p', m\rangle, n\rangle \in add$ is $\mathbf{L^{i}D}_{\lambda}$-deducible.

$$\cfrac{\cfrac{\cfrac{\Rightarrow p\in\mathbf{B}^* \qquad \Rightarrow \langle\langle p', m\rangle, n\rangle \in add}{\Rightarrow p\in\mathbf{B}^* \sqcap \langle\langle p', m\rangle, n\rangle \in add}}{\Rightarrow \bigvee^{\mathbf{B}^*}z\,(\langle\langle z', m\rangle, n\rangle \in add)}}{\Rightarrow \langle m, n\rangle \in \lambda xy \bigvee^{\mathbf{B}^*}z\,(\langle\langle z', x\rangle, y\rangle \in add)}$$

QED

PROPOSITION 2.23. *If* n *is a natural number, then sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.23i) $\quad t\in\mathbf{B}^* \Rightarrow \langle 0, t'\rangle \in less$

(2.23ii) $\quad s\in\mathbf{B}^*, \langle\langle s'', n\rangle, t\rangle \in add \Rightarrow \langle n', t\rangle \in less$

(2.23iii)	$\langle n, s\rangle \in less \Rightarrow n^{\prime} \equiv s \diamond \langle n^{\prime}, s\rangle \in less$
(2.23iv)	$n \equiv s \Rightarrow \langle s, n^{\prime}\rangle \in less$
(2.23v)	$\langle s, 0\rangle \in less \Rightarrow$
(2.23vi)	$\langle s, n^{\prime}\rangle \in less \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n$
(2.23vii)	$\langle s, n\rangle \in less \Rightarrow \langle s, n^{\prime}\rangle \in less$

Proof. *Re* 2.23i. Employ 2.10i:

$$\begin{array}{c}
t \in \mathbf{B}^{*} \Rightarrow t \in \mathbf{B}^{*} \qquad \Rightarrow \langle\langle t^{\prime}, 0\rangle, t^{\prime}\rangle \in add \\
\hline
t \in \mathbf{B}^{*} \Rightarrow t \in \mathbf{B}^{*} \,\square\, \langle\langle t^{\prime}, 0\rangle, t^{\prime}\rangle \in add \\
\hline
t \in \mathbf{B}^{*} \Rightarrow \bigvee^{\mathbf{B}^{*}} z\, (\langle\langle z^{\prime}, 0\rangle, t^{\prime}\rangle \in add) \\
\hline
t \in \mathbf{B}^{*} \Rightarrow \langle 0, t^{\prime}\rangle \in less
\end{array}$$

Re 2.23ii. Employ 2.15i:

$$\begin{array}{c}
s \in \mathbf{B}^{*}, \langle\langle s^{\prime\prime}, n\rangle, t\rangle \in add \Rightarrow s \in \mathbf{B}^{*} \,\square\, \langle\langle s^{\prime}, n^{\prime}\rangle, t\rangle \in add \\
\hline
s \in \mathbf{B}^{*}, \langle\langle s^{\prime\prime}, n\rangle, t\rangle \in add \Rightarrow \bigvee^{\mathbf{B}^{*}} z\, (\langle\langle z^{\prime}, n^{\prime}\rangle, t\rangle \in add) \\
\hline
s \in \mathbf{B}^{*}, \langle\langle s^{\prime\prime}, n\rangle, t\rangle \in add \Rightarrow \langle n^{\prime}, t\rangle \in less
\end{array}$$

Re 2.23iii. Employ an induction on n. As regards the induction basis, employ 2.10iii and 2.23ii. Let $\mathcal{C}$ stand for $0^{\prime} \equiv s \diamond \langle 0^{\prime}, s\rangle \in less$:

$$\begin{array}{c}
\begin{array}{cc}
\dfrac{\langle\langle 0^{\prime}, 0\rangle, s\rangle \in add \Rightarrow 0^{\prime} \equiv s}{\langle\langle 0^{\prime}, 0\rangle, s\rangle \in add \Rightarrow \mathcal{C}} &
\dfrac{a \in \mathbf{B}^{*}, \langle\langle a^{\prime\prime}, 0\rangle, s\rangle \in add \Rightarrow \langle 0^{\prime}, s\rangle \in less}{a \in \mathbf{B}^{*}, \langle\langle a^{\prime\prime}, 0\rangle, s\rangle \in add \Rightarrow \mathcal{C}}
\end{array} \\
\hline
c \in \mathbf{B}^{*}, \langle\langle c^{\prime}, 0\rangle, s\rangle \in add \Rightarrow 0^{\prime} \equiv s \diamond \langle 0^{\prime}, s\rangle \in less \\
\hline
c \in \mathbf{B}^{*} \,\square\, \langle\langle c^{\prime}, 0\rangle, s\rangle \in add \Rightarrow 0^{\prime} \equiv s \diamond \langle 0^{\prime}, s\rangle \in less \\
\hline
\bigvee^{\mathbf{B}^{*}} z\, (\langle\langle z^{\prime}, 0\rangle, s\rangle \in add) \Rightarrow 0^{\prime} \equiv s \diamond \langle 0^{\prime}, s\rangle \in less \\
\hline
\langle 0, s\rangle \in \lambda xy \bigvee^{\mathbf{B}^{*}} z\, (\langle\langle z^{\prime}, x\rangle, y\rangle \in add) \Rightarrow 0^{\prime} \equiv s \diamond \langle 0^{\prime}, s\rangle \in less
\end{array}$$

As regards the induction step, employ again 2.10iii and 2.23ii. Let $\mathcal{C}$ stand for $n^{\prime\prime} \equiv s \diamond \langle n^{\prime\prime}, s\rangle \in less$:

$$\begin{array}{c}
\begin{array}{c} \langle\langle 0', n'\rangle, s\rangle \in add \Rightarrow n'' \equiv s \\ \hline \langle\langle 0', n'\rangle, s\rangle \in add \Rightarrow \mathcal{C} \end{array}
\qquad
\begin{array}{c} a \in \mathbf{B}^*, \langle\langle a'', n'\rangle, s\rangle \in add \Rightarrow \langle n'', s\rangle \in less \\ \hline a \in \mathbf{B}^*, \langle\langle a'', n'\rangle, s\rangle \in add \Rightarrow \mathcal{C} \end{array} \\ \hline
c \in \mathbf{B}^*, \langle\langle c', n'\rangle, s\rangle \in add \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less \\ \hline
c \in \mathbf{B}^* \,\Box\, \langle\langle c', n'\rangle, s\rangle \in add \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less \\ \hline
\bigvee^{\mathbf{B}^*} z_1 \, (\langle\langle z_1{}', n'\rangle, s\rangle \in add) \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less \\ \hline
\langle n', s\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z \, (\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less
\end{array}$$

Re 2.23v.

$$\begin{array}{c}
\begin{array}{c} 0 \equiv c' \Rightarrow \\ \hline s \equiv 0 \wedge 0 \equiv c' \Rightarrow \end{array}
\qquad
\begin{array}{c} 0 \equiv b' \Rightarrow \\ \hline\hline s \equiv a' \wedge 0 \equiv b' \wedge \langle\langle c', a\rangle, b\rangle \in add \Rightarrow \\ \hline\hline \bigvee y \bigvee z \, (s \equiv y' \,\Box\, 0 \equiv z' \,\Box\, \langle\langle c', y\rangle, z\rangle \in add) \Rightarrow \end{array} \\ \hline
(s \equiv 0 \,\Box\, 0 \equiv c') \diamond \bigvee y \bigvee z \, (s \equiv y' \,\Box\, 0 \equiv z' \,\Box\, \langle\langle c', y\rangle, z\rangle \in add) \Rightarrow \\ \hline
\langle\langle c', s\rangle, 0\rangle \in add \Rightarrow \\ \hline
c \in \mathbf{B}^*, \langle\langle c', s\rangle, 0\rangle \in add \Rightarrow \\ \hline
c \in \mathbf{B}^* \,\Box\, \langle\langle c', s\rangle, 0\rangle \in add \Rightarrow \\ \hline
\bigvee^{\mathbf{B}^*} z \, (\langle\langle z', s\rangle, 0\rangle \in add) \Rightarrow \\ \hline
\langle s, 0\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z \, (\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow
\end{array}$$

Re 2.23vi. Employ 2.15ii:

$$\begin{array}{c}
\langle\langle c', s\rangle, n'\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\ \hline
c \in \mathbf{B}^*, \langle\langle c', s\rangle, n\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\ \hline
c \in \mathbf{B}^* \,\Box\, \langle\langle c', s\rangle, n\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\ \hline
\bigvee^{\mathbf{B}^*} z \, (\langle\langle z', s\rangle, n'\rangle \in add) \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\ \hline
\langle s, n'\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z \, (\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n
\end{array}$$

Re 2.23vii. Distinguish two cases according to whether $\mathrm{n} = 0$ or $\mathrm{n} = \mathrm{p}'$, $\mathrm{p} \in \mathbb{N}$. The first case is an immediate consequence of 2.23v. As regards the second case, employ 2.23vi and 2.22:

$$\dfrac{\langle s, p^{\prime}\rangle \in less \Rightarrow s \equiv 0 \diamond \cdots \diamond s \equiv p \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \langle 0, p^{\prime\prime}\rangle \in less}{s \equiv 0 \Rightarrow \langle s, p^{\prime\prime}\rangle \in less} \quad \dfrac{\Rightarrow \langle 0^{\prime}, p^{\prime\prime}\rangle \in less}{s \equiv 0^{\prime} \Rightarrow \langle s, p^{\prime\prime}\rangle \in less}}{s \equiv 0 \diamond s \equiv 0^{\prime} \Rightarrow \langle s, p^{\prime\prime}\rangle \in less}}{\text{p analogous } \diamond\text{-introductions}}}{s \equiv 0 \diamond \cdots \diamond s \equiv p \Rightarrow \langle s, p^{\prime\prime}\rangle \in less}}{\langle s, p^{\prime}\rangle \in less \Rightarrow \langle s, p^{\prime\prime}\rangle \in less} \clubsuit$$

QED

Proposition 2.24. *For all natural numbers* n*, sequents according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible.*

(2.24i) $s \in \mathbf{B}^{*} \Rightarrow \langle s, 0\rangle \in less \diamond s \equiv 0 \diamond \langle 0, s\rangle \in less$

(2.24ii) $\langle s, n\rangle \in less \diamond s \equiv n \diamond \langle n, s\rangle \in less \Rightarrow$
$\langle s, n^{\prime}\rangle \in less \diamond s \equiv n^{\prime} \diamond \langle n^{\prime}, s\rangle \in less$

(2.24iii) $s \in \mathbf{B}^{*} \Rightarrow \langle s, n\rangle \in less \diamond s \equiv n \diamond \langle n, s\rangle \in less$

Proof. Re 2.24i. Employ 2.23i:

$$\dfrac{\dfrac{\Rightarrow 0 \equiv 0}{\Rightarrow \langle 0, 0\rangle \in less \diamond 0 \equiv 0 \diamond \langle 0, 0\rangle \in less} \quad \dfrac{a \in \mathbf{B}^{*} \Rightarrow \langle 0, a^{\prime}\rangle \in less}{a \in \mathbf{B}^{*} \Rightarrow \langle a^{\prime}, 0\rangle \in less \diamond a^{\prime} \equiv 0 \diamond \langle 0, a^{\prime}\rangle \in less}}{s \in \mathbf{B}^{*} \Rightarrow \langle s, 0\rangle \in less \diamond s \equiv 0 \diamond \langle 0, s\rangle \in less}.$$

Re 2.24ii. By 2.23vii

$$\dfrac{\langle s, n\rangle \in less \Rightarrow \langle s, n^{\prime}\rangle \in less}{\langle s, n\rangle \in less \Rightarrow \langle s, n^{\prime}\rangle \in less \diamond s = n^{\prime} \diamond \langle n^{\prime}, s\rangle \in less},$$

and by 2.23iv

$$\dfrac{s \equiv n \Rightarrow \langle s, n^{\prime}\rangle \in less}{\langle s, n\rangle \in less \Rightarrow \langle s, n^{\prime}\rangle \in less \diamond s = n^{\prime} \diamond \langle n^{\prime}, s\rangle \in less}.$$

and by 2.23iii

$$\dfrac{\langle n, s\rangle \in less \Rightarrow n^{\prime} \equiv s \diamond \langle n^{\prime}, s\rangle \in less}{\langle s, n\rangle \in less \Rightarrow \langle s, n^{\prime}\rangle \in less \diamond s = n^{\prime} \diamond \langle n^{\prime}, s\rangle \in less}.$$

Together:

$$\langle s, n\rangle \in less \diamond s \equiv n \diamond \langle n, s\rangle \in less \Rightarrow \langle s, n^{\prime}\rangle \in less \diamond s \equiv n^{\prime} \diamond \langle n^{\prime}, s\rangle \in less$$

Re 2.24iii. Employing 2.24i and 2.24ii for a metatheoretical induction gives

$$s\in\mathbf{B}^* \Rightarrow \langle s,n\rangle\in less \diamond s\equiv n \diamond \langle n,s\rangle\in less$$

for all natural numbers n. QED

COROLLARY 2.25. *Inferences according to the following schema are* $\mathbf{L^iD}_\lambda$*-derivable*

$$\dfrac{\langle s,n\rangle\in less \diamond s\equiv n\diamond\langle n,s\rangle\in less, \Gamma\Rightarrow C}{s\in\mathbf{B}^*, \Gamma\Rightarrow C}$$

PROPOSITION 2.26. *If* m *is a natural number,* n⃗ *a* p*-tuple of natural numbers and* $\mathfrak{C} :\equiv \langle\langle *_1, *_2\rangle, 0\rangle\in s \sqcap \bigwedge z\,(\langle z,*_2\rangle\in less \to \langle\langle *_1, z\rangle,0\rangle\notin s)$*, then sequents according to the following schemata are* $\mathbf{L^iD}_\lambda$*-deducible.*

(2.26i) $\langle\langle\vec{n},m\rangle,0\rangle\in s, m\equiv 0\Rightarrow\mathfrak{C}[\vec{n},m]$

(2.26ii) $\langle\langle\vec{n},c\rangle,0\rangle\in s, \langle c,k^{\prime}\rangle\in less\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s$

(2.26iii) $\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle m,c\rangle\in less, \mathfrak{C}[\vec{n},m]\Rightarrow$

Proof. *Re* 2.26i. Employ 2.23v:

$$\dfrac{\langle\langle\vec{n},m\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},m\rangle,0\rangle\in s \qquad \dfrac{\dfrac{\dfrac{\dfrac{\langle a,0\rangle\in less\Rightarrow}{m\equiv 0,\langle a,m\rangle\in less\Rightarrow}}{m\equiv 0,\langle a,m\rangle\in less\Rightarrow\langle\langle\vec{n},a\rangle,0\rangle\notin s)}}{m\equiv 0\Rightarrow\langle a,m\rangle\in less\to\langle\langle\vec{n},a\rangle,0\rangle\notin s)}}{m\equiv 0\Rightarrow\bigwedge z_1\,(\langle z_1,m\rangle\in less\to\langle\langle\vec{n},z_1\rangle,0\rangle\notin s)}}{\langle\langle\vec{n},m\rangle,0\rangle\in s, m\equiv 0\Rightarrow\langle\langle\vec{n},m\rangle,0\rangle\in s\sqcap\bigwedge z_1\,(\langle z_1,m\rangle\in less\to\langle\langle\vec{n},z_1\rangle,0\rangle\notin s)}\ .$$

Re 2.26ii. Let $\mathcal{A}$ stand for $c\equiv 0\diamond\ldots\diamond c\equiv k$, k $\geq$ 1. Employ 2.23vi:

$$\dfrac{\langle c,k^{\prime}\rangle\in less\Rightarrow\mathcal{A} \qquad \dfrac{\dfrac{\dfrac{c\equiv 0,\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s \qquad c\equiv 0^{\prime},\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0^{\prime}\rangle,0\rangle\in s}{c\equiv 0\diamond c\equiv 0^{\prime},\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s}}{\text{k} - 1 \text{ analogous } \diamond\text{-introductions left}}}{\mathcal{A},\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s}}{\langle\langle\vec{n},c\rangle,0\rangle\in s,\langle c,k^{\prime}\rangle\in less\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s}\ \clubsuit\ .$$

Re 2.26iii.

$$\dfrac{\dfrac{\dfrac{\langle m,c\rangle\in less \Rightarrow \langle m,c\rangle\in less \qquad \dfrac{\langle\langle\vec{n},m\rangle,0\rangle\in s \Rightarrow \langle\langle\vec{n},m\rangle,0\rangle\in s}{\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle\langle\vec{n},m\rangle,0\rangle\notin s \Rightarrow}}{\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle m,c\rangle\in less, \langle m,c\rangle\in less \rightarrow \langle\langle\vec{n},m\rangle,0\rangle\notin s \Rightarrow}}{\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle m,c\rangle\in less, \bigwedge z\,(\langle z,c\rangle\in less \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin s) \Rightarrow}}{\dfrac{\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle m,c\rangle\in less, \langle\langle\vec{n},m\rangle,0\rangle\in s, \bigwedge z\,(\langle z,c\rangle\in less \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin s) \Rightarrow}{\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle m,c\rangle\in less, \mathfrak{C}[\vec{n},m] \Rightarrow}}$$

QED

DEFINITION 2.27.
$min[s] :\equiv \lambda xy\,(y\in\mathbf{B}^* \Box \langle\langle\vec{x},y\rangle,0\rangle\in s \Box \bigwedge z\,(\langle z,y\rangle\in less \rightarrow \langle\langle\vec{x},z\rangle,0\rangle\notin s))$.

PROPOSITION 2.28. *If the function* $\mathfrak{g}(\vec{\mathrm{x}},\mathrm{y})$ *is numeralwise represented in* $\mathbf{L^iD}_\lambda$ *by the term* g, *and the function* $\mathfrak{f}(\vec{\mathrm{x}}) = \mu\mathrm{y}\,(\mathfrak{g}(\vec{\mathrm{x}},\mathrm{y}) = 0)$ *obtained from* $\mathfrak{g}$ *by* μ*-recursion is total, then* $\mathfrak{f}$ *is numeralwise represented in* $\mathbf{L^iD}_\lambda$ *by* $min[g]$.

Proof. If $\mathfrak{g}(\vec{\mathrm{n}},\mathrm{m}) = 0$ and $\mathfrak{f}(\vec{\mathrm{n}}) = \mathrm{m}$, *i.e.*, $\mu\mathrm{y}\,(\mathrm{g}(\vec{\mathrm{n}},\mathrm{y}) = 0) = \mathrm{m}$, then by the assumption that $\mathfrak{g}$ is numeralwise represented in $\mathbf{L^iD}_\lambda$ by g, we have that

(2.28i) $\quad \Rightarrow \langle\langle\vec{n},m\rangle,0\rangle\in g,$ and

(2.28ii) $\quad \Rightarrow \bigwedge x\,(\langle\langle\vec{n},m\rangle,x\rangle\in g \rightarrow x\equiv 0)$

(2.28iii) $\quad \langle\langle\vec{n},i\rangle,0\rangle\in g \Rightarrow \qquad$ if for all $\mathrm{i} < \mathrm{m}$

are $\mathbf{L^iD}_\lambda$-deducible. In addition, 2.28iii yields:

(2.28iv) $\quad \langle\langle\vec{n},0\rangle,0\rangle\in g \diamond \cdots \diamond \langle\langle\vec{n},k\rangle,0\rangle\in g \Rightarrow$

for $\mathrm{k}' = \mathrm{m}$ by successive $\diamond$-introduction. (There is no $\mathrm{i} < \mathrm{m}$ for $\mathrm{m} = 0$.)

Now, what has to be shown for the numeralwise representability of minimization is that

$$\Rightarrow \langle\vec{n},m\rangle\in min[g], \quad \text{and}$$
$$\Rightarrow \bigwedge x\,(\langle\vec{n},x\rangle\in min[g] \rightarrow x\equiv m)\,.$$

are $\mathbf{L^iD}_\lambda$-deducible. First of all, cutting 2.28i with 2.26i yields

$$m\equiv 0 \Rightarrow \bigwedge z_1\,(\langle z_1,m\rangle\in less \rightarrow \langle\langle\vec{n},z_1\rangle,0\rangle\notin s)\,, \quad \text{and}$$

cutting 2.26ii and 2.28iv gives way to the following deduction:

$$\begin{array}{c}
\langle\langle\vec{n},c\rangle,0\rangle\in g, \langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow \\ \hline
\langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow \langle\langle\vec{n},c\rangle,0\rangle\notin g \\ \hline
\langle c,m\rangle\in \mathit{less}, m\equiv k^{\prime} \Rightarrow \langle\langle\vec{n},c\rangle,0\rangle\notin g \\ \hline
m\equiv k^{\prime} \Rightarrow \langle c,m\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},c\rangle,0\rangle\notin g \\ \hline
m\equiv k^{\prime} \Rightarrow \bigwedge z\,(\langle z,m\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin g)
\end{array}$$

This yields

$$m\equiv 0 \Rightarrow \langle\vec{n},m\rangle\in \mathit{min}[g]\,, \quad \text{and}$$

$$m\equiv k^{\prime} \Rightarrow \langle\vec{n},m\rangle\in \mathit{min}[g]$$

in the following way (where Γ is $m\equiv 0$, $m\equiv k^{\prime}$, resp.), employing 2.28i:

$$\begin{array}{c}
\Rightarrow \langle\langle\vec{n},m\rangle,0\rangle\in g \qquad \Gamma \Rightarrow \bigwedge z\,(\langle z,m\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin g) \\ \hline
\Rightarrow m\in\mathbf{B}^{*} \qquad \Gamma \Rightarrow \langle\langle\vec{n},m\rangle,0\rangle\in g \,\Box\, \bigwedge z\,(\langle z,m\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin g) \\ \hline\hline
\Gamma \Rightarrow m\in\mathbf{B}^{*} \,\Box\, \langle\langle\vec{n},m\rangle,0\rangle\in g \,\Box\, \bigwedge z\,(\langle z,m\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin g)) \\ \hline\hline
\Gamma \Rightarrow \langle\vec{n},m\rangle\in\lambda\vec{x}y\,(y\in\mathbf{B}^{*} \,\Box\, \langle\langle\vec{x},y\rangle,0\rangle\in g \,\Box\, \bigwedge z\,(\langle z,y\rangle\in \mathit{less} \rightarrow \langle\langle\vec{x},z\rangle,0\rangle\notin g))
\end{array}$$

Cutting 2.26ii and 2.28iv gives way to the following deduction, where $\mathfrak{C}$ is the nominal form $\langle\langle\vec{n},*_1\rangle,0\rangle\in g$:

$$\begin{array}{c}
\langle\langle\vec{n},c\rangle,0\rangle\in g, \langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow \mathfrak{C}[0]\diamond\cdots\diamond\mathfrak{C}[k] \qquad \mathfrak{C}[0]\diamond\cdots\diamond\mathfrak{C}[k] \Rightarrow \\ \hline\hline
\langle\langle\vec{n},c\rangle,0\rangle\in g, \langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow \\ \hline\hline
m\in\mathbf{B}^{*}, \langle\langle\vec{n},c\rangle,0\rangle\in g, \bigwedge z\,(\langle z,c\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin s), \langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow \\ \hline\hline
m\in\mathbf{B}^{*} \,\Box\, \langle\langle\vec{n},c\rangle,0\rangle\in g \,\Box\, \bigwedge z\,(\langle z,c\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin s), \langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow \\ \hline
\langle\vec{n},c\rangle\in \mathit{min}[g], \langle c,k^{\prime}\rangle\in \mathit{less} \Rightarrow
\end{array} \;\clubsuit$$

Since

$$\langle\vec{n},c\rangle\in \mathit{min}[g], \langle c,0\rangle\in \mathit{less} \Rightarrow$$

holds almost trivially as a consequence of

$$\langle c,0\rangle\in \mathit{less} \Rightarrow \qquad (2.23\text{v})\,,$$

and m is either 0 or k′ for some natural number k, we have that

$$\langle\vec{n},c\rangle\in \mathit{min}[g], \langle c,m\rangle\in \mathit{less} \Rightarrow$$

is $\mathbf{L^{i}D}_{\lambda}$-deducible.

The last preparational step is to provide

$$\langle\vec{n}, c\rangle \in min[g], \langle m, c\rangle \in less \Rightarrow \quad .$$

which is readily obtained from 2.28ii and 2.26iii by means of a cut.

This now yields the second condition of numeralwise representability of minimization in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{c \equiv m \Rightarrow c \equiv m \qquad \dfrac{\langle\vec{n}, c\rangle \in min[g], \langle c, m\rangle \in less \Rightarrow \qquad \langle\vec{n}, c\rangle \in min[g], \langle m, c\rangle \in less \Rightarrow}{\langle\vec{n}, c\rangle \in min[g], \langle c, m\rangle \in less \diamond \langle m, c\rangle \in less \Rightarrow}}{\langle\vec{n}, c\rangle \in min[g], \langle c, m\rangle \in less \diamond \langle m, c\rangle \in less \diamond c \equiv m \Rightarrow c \equiv m}\;2.25}{\langle\vec{n}, c\rangle \in min[g] \Rightarrow c \equiv m}}{\dfrac{\Rightarrow \langle\vec{n}, c\rangle \in min[g] \rightarrow c \equiv m}{\Rightarrow \bigwedge x\, (\langle\vec{n}, x\rangle \in min[g] \rightarrow x \equiv m)}}\ .$$

QED

THEOREM 2.29. *The recursive functions are numeralwise representable in* $\mathbf{L^iD}_\lambda$.

Proof. As for result 45.46 in [**15**], p. 573, this is an immediate consequence of the numeralwise representability of addition, multiplication, the identity functions, the characteristic function of equality, composition, and minimization. QED

THEOREM 2.30. $\mathbf{L^iD}_\lambda$ *is essentially undecidable.*

Proof. As for any consistent theory which allows numeralwise representability of all recursive functions.[10] QED

REMARK 2.31. In view of the cut eliminability in $\mathbf{L^iD}_\lambda$, the foregoing two results extend to $\mathbf{LP}_\lambda$.

3. Addition 130f. Fixed points and denotational devices

Definite description can only be established in a somewhat reduced form in $\mathbf{L^iD}_\lambda$, the reason being a contraction that sneaks into the proof of proposition 41.17 in [**15**], p. 470. Could this contraction possibly do harm? In the present section I shall show that it actually does.

[10] Cf. theorem 48.27 in [**15**], p. 612, for the paradigm of proof.

The failure of extensionality may already be seen as an indication that denotation is not quite as straightforward a business as was thought in the early days of modern logic. The following application of the fixed point property to establish the incompatibility of indefinite description (ε-operator) with $\mathbf{L^{i}D}_{\lambda}$ may well be seen as contributing to this view.

PROPOSITION 3.1. $\mathbf{L^{i}D}_{\lambda} \cup \{\bigvee x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[\varepsilon x\, \mathfrak{F}[x]]\} \vdash \bot$

Proof. Take the fixed point $\phi = \varepsilon x\,(\phi \neq x)$ and consider the following deduction:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\bigvee x\,(\phi \neq x) \Rightarrow \phi \neq \varepsilon x\,(\phi \neq x)
\qquad
\dfrac{\Rightarrow \phi = \varepsilon x\,(\phi \neq x)}{\phi \neq \varepsilon x\,(\phi \neq x) \Rightarrow}
}{\bigvee x\,(\phi \neq x) \Rightarrow}\clubsuit
}{\phi \neq 0 \Rightarrow}
}{\phi = 1, 1 \neq 0 \Rightarrow}
}{\phi = 1 \Rightarrow}
}{\Rightarrow \phi \neq 1}
\qquad
\dfrac{
\begin{matrix}\text{as on the left}\\ \vdots \\ \bigvee x\,(\phi \neq x) \Rightarrow\end{matrix}
}{\phi \neq 1 \Rightarrow}
}{\Rightarrow}\clubsuit
$$

QED

REMARK 3.2. Notice that there is no contraction involved in this deduction. They are hiding in the ε-initial sequent.[11]

That the ε-operator is not compatible with $\mathbf{L^{i}D}_{\lambda}$ may not surprise people who find the ε-operator outrageous anyway; so I shall show that the least number operator doesn't fare any better.

First of all: the formulation of the least number operator has to be restricted to natural numbers. But in view of the fact that only 0 and 1 are employed in the proof above, this is little more than a formality. The definition of the natural numbers provided in 41.60 on p. 487 of [**15**] would actually do, since even without contraction it still yields 0 and 1 as natural numbers.

PROPOSITION 3.3. $\mathbf{L^{i}D}_{\lambda} \cup \{\bigvee^{\mathrm{N}} x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[\mu x\, \mathfrak{F}[x]]\} \vdash \bot$.

[11] This has been used as a convenient way of "proving" that abandoning contraction is no safeguard against the paradoxes.

Proof. Take the fixed point $\phi = \mu x\,(\phi \neq x)$ and consider the following deduction:

$$
\dfrac{
  \dfrac{
    \dfrac{\Rightarrow 0\in\mathbf{N} \qquad
      \dfrac{\dfrac{\dfrac{\bigvee^{\mathrm{N}}x\,(\phi \neq x) \Rightarrow \phi \neq \mu x\,(\phi \neq x) \qquad \dfrac{\Rightarrow \phi = \mu x\,(\phi \neq x)}{\phi \neq \mu x\,(\phi \neq x) \Rightarrow}}{\bigvee^{\mathrm{N}}x\,(\phi \neq x) \Rightarrow}\clubsuit}{0\in\mathbf{N} \sqcap \phi \neq 0 \Rightarrow}}{0\in\mathbf{N}, \phi \neq 0 \Rightarrow}
    }{\dfrac{\dfrac{\phi \neq 0 \Rightarrow}{\phi = 1, 1 \neq 0 \Rightarrow}}{\dfrac{\phi = 1 \Rightarrow}{\Rightarrow \phi \neq 1}}}\clubsuit
  }{} \qquad
  \dfrac{\Rightarrow 1\in\mathbf{N} \qquad \dfrac{\dfrac{\begin{array}{c}\text{as on the left}\\ \vdots \\ \bigvee^{\mathrm{N}}x\,(\phi \neq x) \Rightarrow\end{array}}{1\in\mathbf{N} \sqcap \phi \neq 1 \Rightarrow}}{1\in\mathbf{N}, \phi \neq 1 \Rightarrow}}{\phi \neq 1 \Rightarrow}\clubsuit
}{\Rightarrow}\clubsuit
$$

QED

Still, someone who objects to the least number operator on the basis of the non-finite character of the least number principle[12] might not find that too surprising. So I shall go one step further and show that even without the least number principle one obtains a contradiction.

Proposition 3.4.

$$\mathbf{L^{i}D}_{\lambda} \cup \{\bigvee^{\mathrm{N}}x\,(\mathfrak{F}[x] \sqcap \bigwedge^{\mathrm{N}}y\,(y < x \rightarrow \neg\mathfrak{F}[y])) \Rightarrow \mathfrak{F}[\mu x\,\mathfrak{F}[x]]\} \vdash \bot\,.$$

Proof. Take again the fixed point $\phi = \mu x\,(\phi \neq x)$ and consider the following variation of the by now familiar deduction which yields $\bigvee^{\mathrm{N}}x\,(\phi \neq x \sqcap \bigwedge^{\mathrm{N}}y\,(y < x \rightarrow \neg(\phi \neq y)))) \Rightarrow$ instead of $\bigvee^{\mathrm{N}}x\,(\phi \neq x) \Rightarrow$ and continue as follows:

$$
\dfrac{\Rightarrow 0\in\mathbf{N} \qquad
  \dfrac{
    \dfrac{\dfrac{b\in\mathbf{N}, b < 0 \Rightarrow \neg(\phi \neq b)}{b\in\mathbf{N} \Rightarrow b < 0 \rightarrow \neg(\phi \neq b)}}{\Rightarrow \bigwedge^{\mathrm{N}}y\,(y < 0 \rightarrow \neg(\phi \neq y))}
    \qquad
    \dfrac{\dfrac{\bigvee^{\mathrm{N}}x\,(\phi \neq x \sqcap \bigwedge^{\mathrm{N}}y\,(y < x \rightarrow \neg(\phi \neq y)))) \Rightarrow}{0\in\mathbf{N} \sqcap \phi \neq 0 \sqcap \bigwedge^{\mathrm{N}}y\,(y < 0 \rightarrow \neg(\phi \neq y)))) \Rightarrow}}{0\in\mathbf{N}, \phi \neq 0, \bigwedge^{\mathrm{N}}y\,(y < 0 \rightarrow \neg(\phi \neq y)))) \Rightarrow}
  }{0\in\mathbf{N}, \phi \neq 0 \Rightarrow}\clubsuit
}{\phi \neq 0 \Rightarrow}\clubsuit
$$

(The inference to $0\in\mathbf{N}, \phi \neq 0, \bigwedge^{\mathrm{N}}y\,(y < 0 \rightarrow \neg(\phi \neq y)))) \Rightarrow$ is marked with a double line.)

---

12 Cf. proposition 46.20 in [**15**] for the least number principle in first order arithmetic. As regards the non-finite character of the least number principle, cf. Bernays in quotation 80.6 in the [**15**].

This result can now be employed to yield $\Rightarrow \phi \neq 1$ as usual, but also to prove $\bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y))))$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{b \in \mathbf{N}, b < 1 \Rightarrow b = 0 \qquad \dfrac{\phi \neq 0 \Rightarrow}{b = 0, \phi \neq b \Rightarrow}}{b \in \mathbf{N}, b < 1, \phi \neq b \Rightarrow}\;\clubsuit}{b \in \mathbf{N}, b < 1 \Rightarrow \phi \neq b}}{b \in \mathbf{N} \Rightarrow b < 1 \rightarrow \phi \neq b}}{\Rightarrow \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y))}$$

Continue as above, only with 1 instead of 0:

$$\dfrac{\Rightarrow 1 \in \mathbf{N} \qquad \dfrac{\Rightarrow \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y)) \qquad \dfrac{\dfrac{\bigvee^{\mathrm{N}} x\,(\phi \neq x \,\square\, \bigwedge^{\mathrm{N}} y\,(y < x \rightarrow \neg(\phi \neq y))) \Rightarrow}{1 \in \mathbf{N} \,\square\, \phi \neq 1 \,\square\, \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y)) \Rightarrow}}{1 \in \mathbf{N}, \phi \neq 1, \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y)))) \Rightarrow}}{1 \in \mathbf{N}, \phi \neq 1 \Rightarrow}\;\clubsuit}{\phi \neq 1 \Rightarrow}\;\clubsuit$$

QED

This result can be extended to the ι-operator with the usual initial sequent. The point is that the least number operator is just a special form of definite description and the least number principle and the only natural numbers actually employed are 0 and 1.

CONVENTION 3.5.
$\mathfrak{C}[s] :\equiv s \in \{0,1\} \,\square\, \mathfrak{F}[s] \,\square\, \bigwedge y\,(y \in \{0,1\} \,\square\, y < s \rightarrow \neg\mathfrak{F}[y])\,.$

PROPOSITION 3.6. *If $\mathfrak{C}[s]$ is according to convention 3.5, then sequents according to the following schemata are $\mathbf{L^{i}D}_{\lambda}$-deducible.*

(3.6i) $a \equiv 0, b \equiv 0 \Rightarrow a = b$

(3.6ii) $a \equiv 1, b \equiv 1 \Rightarrow a = b$

(3.6iii) $a \equiv 0, b \equiv 1, \mathfrak{F}[a], \bigwedge y\,(y \in \{0,1\} \,\square\, y < b \rightarrow \neg\mathfrak{F}[y]) \Rightarrow a = b$

(3.6iv) $a \equiv 1, b \equiv 0, \mathfrak{F}[b], \bigwedge y\,(y \in \{0,1\} \,\square\, y < a \rightarrow \neg\mathfrak{F}[y]) \Rightarrow a = b$

(3.6v) $\mathfrak{C}[a], \mathfrak{C}[b] \Rightarrow a = b$

(3.6vi) $\bigvee x\, \mathfrak{C}[x] \Rightarrow \bigvee x\,(\mathfrak{C}[x] \wedge \bigwedge y\,(\mathfrak{C}[y] \rightarrow x = y))$

(3.6vii) $\bigvee x\, \mathfrak{C}[x] \Rightarrow \bigvee x\, \mathfrak{C}[x] \,\square\, \bigwedge x \bigwedge y\,(\mathfrak{C}[x] \,\square\, \mathfrak{C}[y] \rightarrow x = y)$

Proof. *Re* 3.6i and ii. Trivial.
Re 3.6iii

$$\dfrac{\dfrac{\dfrac{\Rightarrow 0\in\{0,1\} \qquad \Rightarrow 0<1}{\Rightarrow 0\in\{0,1\}\Box 0<1}}{a\equiv 0, b\equiv 1 \Rightarrow a\in\{0,1\}\Box a<b} \qquad \dfrac{\dfrac{\mathfrak{F}[a]\Rightarrow\mathfrak{F}[a]}{\mathfrak{F}[a],\neg\mathfrak{F}[a]\Rightarrow}}{\mathfrak{F}[a],\neg\mathfrak{F}[a]\Rightarrow a=b}}{\dfrac{a\equiv 0, b\equiv 1, \mathfrak{F}[a], a\in\{0,1\}\Box a<b\rightarrow\neg\mathfrak{F}[a]\Rightarrow a=b}{a\equiv 0, b\equiv 1, \mathfrak{F}[a], \bigwedge y\,(y\in\{0,1\}\Box y<b\rightarrow\neg\mathfrak{F}[y])\Rightarrow a=b}}$$

Re 3.6iv. As for 3.6iii; left to the reader.
Re 3.6v. This is straightforward consequence of 3.6i–3.6iv.
Re 3.6vi. Employ 3.6v:

$$\dfrac{\dfrac{\dfrac{\mathfrak{C}[a]\Rightarrow\mathfrak{C}[a] \qquad \dfrac{\dfrac{\mathfrak{C}[a],\mathfrak{C}[b]\Rightarrow a=b}{\mathfrak{C}[a]\Rightarrow\mathfrak{C}[b]\rightarrow a=b}}{\mathfrak{C}[a]\Rightarrow\bigwedge y\,(\mathfrak{C}[y]\rightarrow a=y)}}{\mathfrak{C}[a]\Rightarrow\mathfrak{C}[a]\wedge\bigwedge y\,(\mathfrak{C}[y]\rightarrow a=y)}}{\mathfrak{C}[a]\Rightarrow\bigvee x\,(\mathfrak{C}[x]\wedge\bigwedge y\,(\mathfrak{C}[y]\rightarrow x=y))}}{\bigvee x\,\mathfrak{C}[x]\Rightarrow\bigvee x\,(\mathfrak{C}[x]\wedge\bigwedge y\,(\mathfrak{C}[y]\rightarrow x=y))}$$

Re 3.6vii. Straightforward in view of 3.6v; left to the reader. QED

THEOREM 3.7.

(3.7i) $\mathbf{L^iD}_\lambda\cup\{\bigvee x\,(\mathfrak{F}[x]\wedge\bigwedge y\,(\mathfrak{F}[y]\rightarrow x=y))\Rightarrow\mathfrak{F}[\iota x\,\mathfrak{F}[x]]\}\vdash\bot$

(3.7ii) $\mathbf{L^iD}_\lambda\cup\{\bigvee x\,\mathfrak{F}[x], \bigwedge z_1\bigwedge z_2\,(\mathfrak{F}[z_1]\Box\mathfrak{F}[z_2]\rightarrow z_1=z_2)$
$\Rightarrow\mathfrak{F}[\iota x\,\mathfrak{F}[x]]\}\vdash\bot$

Proof. The point is, of course, to find an appropriate $\mathfrak{F}$. That's what convention 3.5 has been designed for. In view of 3.6vi and 3.6vii, both, 3.7i and 3.7ii, essentially reduce to a form of 3.1, only with ι instead of ε:

$$\bigvee x\,\mathfrak{F}[x]\Rightarrow\mathfrak{F}[\iota x\,\mathfrak{F}[x]]\,,$$

where $\mathfrak{F}:\equiv *_1\in\{0,1\}\Box\phi\neq *_1\Box\bigwedge y\,(y\in\{0,1\}\Box y<*_1\rightarrow\neg(\phi\neq *_1))$ with ϕ being the fixed point satisfying $\phi=\iota x\mathfrak{F}[x]$. Since $\mathfrak{F}[\iota x\mathfrak{F}[x]]\Rightarrow\phi\neq\iota x\mathfrak{F}[x]$ is straightforward, one obtains

$$\bigvee x\,(x\in\{0,1\}\Box\phi\neq x\Box\bigwedge y\,(y\in\{0,1\}\Box y<x\rightarrow\neg(\phi\neq x))\Rightarrow\phi\neq\iota x\,\mathfrak{F}[x]\,.$$

The further procedure is essentially as for 3.4; the fixed point property provides $\phi \neq \iota x\, \mathfrak{F}[x] \Rightarrow$ and with some inversions of $\bigvee$ and $\Box$ in the antecedent this gives:

$$0 \in \{0,1\}, \phi \neq 0, \bigwedge y\, (y \in \{0,1\} \Box\, y < 0 \to \neg(\phi \neq 0)) \Rightarrow \; .$$

As before, one gets $\phi \neq 0 \Rightarrow$ and thereby $\phi \neq 1 \Rightarrow$ which, in turn, yields

$$\Rightarrow \bigwedge y\, (y \in \{0,1\} \Box\, y < 1 \to \neg(\phi \neq 1))$$

as in the proof of 3.4; together with $\Rightarrow 1 \in \{0,1\}$ and

$$1 \in \{0,1\}, \phi \neq 1, \bigwedge y\, (y \in \{0,1\} \Box\, y < 1 \to \neg(\phi \neq 1)) \Rightarrow$$

one obtains $\phi \neq 1 \Rightarrow$ by cut, hence a contradiction. QED

4. Addition 135g: An interpretation of $\lambda\beta$ in $\mathbf{L^iD^Z_\lambda}$

The availability of a notion of weak implication accounts for the possibility of expressing an arbitrary number of simple substitutions, *i.e.*, of substitutions achieved on the basis of $s = t$. This, in turn, makes it possibile to interpret $\lambda\beta$ in $\mathbf{L^iD^Z_\lambda}$.

DEFINITIONS 4.1. (1) $s \tilde{\in} t :\equiv \bigwedge y\, (s = y \supset y \in t)$.
(2) $\lambda^\dagger xy\, \mathfrak{F}[x,y] :\equiv \lambda z \bigwedge x_1 \bigwedge y\, (z = \langle x_1, y \rangle \supset x_1 \in \lambda x\, \mathfrak{F}[x,y])$.
(3) LD-translation of λ-terms and wffs.

$$\text{(3.1)}\quad \mathfrak{A}[\|x\|^{\mathrm{LD}}] \quad :\equiv \begin{cases} \mathfrak{A}[a] & \text{iff } x \text{ is not bound in } \mathfrak{A}\text{, and } a \text{ is the} \\ & \text{first in the list of free variables that} \\ & \text{does not occur in } \mathfrak{A} \\ \mathfrak{A}[x] & \text{otherwise} \end{cases}$$

(3.2) $\|\lambda x.\, A\|^{\mathrm{LD}} \quad :\equiv \lambda^\dagger xy\, (y = \|A\|^{\mathrm{LD}})$
(3.3) $\|AB\|^{\mathrm{LD}} \quad :\equiv \lambda x \bigwedge y\, (\langle \|B\|^{\mathrm{LD}}, y \rangle \tilde{\in} \|A\|^{\mathrm{LD}} \to x \in y)$
(3.4) $\|A = B\|^{\mathrm{LD}} \quad :\equiv \|A\|^{\mathrm{LD}} = \|B\|^{\mathrm{LD}}$

where y does not occur in A in clause (3.2), and neither x nor y occurs in AB in clause (3.3).

CONVENTION 4.2. For the sake of simplicity, I shall write $\|A\|$ instead of $\|A\|^{\mathrm{LD}}$ for the remainder of this section.

EXAMPLES 4.3. The following examples are meant to give an idea of how λ-terms look under the LD-translation.

(1) $\|\lambda x\,.\,x\| \equiv \lambda z \bigwedge x_1 \bigwedge y\,(z = \langle x_1, y\rangle \supset x_1 \in \lambda(y = x))\,.$

(2) $\|\lambda xy\,.\,x\| \equiv \lambda z_1 \bigwedge x_1 \bigwedge y_1\,(z_1 = \langle x_1, y_1\rangle \supset$
$x_1 \in \lambda(y_1 = \lambda z_2\,(\bigwedge x_2 \bigwedge y_2\,(z_2 = \langle x_2, y_2\rangle \supset x_2 \in \lambda(y_2 = x)))))\,.$

(3) $\|\lambda xy\,.\,xyy\| \equiv \lambda^{\dagger} x z_1\,(z_1 = \lambda^{\dagger} y z_2\,(z_2 =$
$\lambda x_1 \bigwedge y_1\,(\langle y, y_1\rangle \tilde{\in} \lambda x_2 \bigwedge y_2\,(\langle y, y_2\rangle \tilde{\in} x_2 \to x_2 \in y_2)) \to x_1 \in y_1))\,.$

PROPOSITION 4.4. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda$*-derivable.*

(4.4i)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[\|x\|]}{\Gamma \Rightarrow \bigwedge x\, \mathfrak{F}[x]}$$

if $\|x\|$ *does not occur in the lower sequent.*

(4.4ii)
$$\frac{\mathfrak{F}[\|A\|], \Gamma \Rightarrow C}{\bigwedge x\, \mathfrak{F}[x], \Gamma \Rightarrow C}$$

Proof. Re 4.4i. This is a straightforward consequence of clause (3.1) of definition 4.1.
Re 4.4ii. This is obvious in view of the fact that $\|A\|$ is a term in the language of $\mathbf{L^iD}_\lambda$. QED

PROPOSITION 4.5. *If the bound variable* x *does not occur in* $\mathfrak{F}$*, then*

$$\|B\| \in \lambda x\, \mathfrak{F}[x, \|A_1\|, \ldots, \|A_{\mathrm{n}}\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|A_1[x/B]\|, \ldots, \|A_{\mathrm{n}}[x/B]\|]\,.$$

is $\mathbf{L^iD}_\lambda$*-deducible.*

Proof. Employ an induction on the sum of the lengths of the A_{i}, where $\mathrm{i} \in \{1, \ldots, \mathrm{n}\}$. To save space, I confine myself to $\mathrm{n} = 1$. Distinguish cases according to the clauses of definition 42.11 in [**15**], p. 502.
1. $A \equiv x$, *i.e.*, what has to be shown is

$$\mathbf{L^iD}^{\mathrm{Z}}_\lambda \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|x\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|x[x/B]\|]\,.$$

By definition 4.1 (3.1) and 42.11 (1) in [**15**], this amounts to showing

$$\mathbf{L^iD}^{\mathrm{Z}}_\lambda \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, x] \Leftrightarrow \mathfrak{F}[\|B\|, \|B\|]\,,$$

which is a straightforward application of λ-abstraction.

2. $A \equiv y$, where $x \not\equiv y$, *i.e.*, what has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|y\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|y[x/B]\|]\,.$$

By definition 4.1 (3.1) and 42.11 (2) in **[15]**, this amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|y\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|y\|]\,,$$

which is a straightforward application of λ-abstraction.
3. $A \equiv (C_1C_2)$, *i.e.*, what has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(C_1C_2)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(C_1C_2)[x/B]\|]\,.$$

By definition 42.11 (3) in **[15]**, this amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(C_1C_2)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(C_1[x/B]C_2[x/B])\|]\,,$$

which, by definition 4.1 (3.3), amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \lambda x_1 \bigwedge y_1\, (\langle \|C_2\|, y_1\rangle \,\tilde{\in}\, \|C_1\| \rightarrow x_1 \in y_1)] \Leftrightarrow$$
$$\mathfrak{F}[\|B\|, \lambda x_1 \bigwedge y_1\, (\langle \|C_2[x/B]\|, y_1\rangle \,\tilde{\in}\, \|C_1[x/B]\|) \rightarrow x_1 \in y_1)]\,,$$

which, in turn, follows by the induction hypothesis.
4. $A \equiv (\lambda x\,.\,C)$. What has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda x\,.\,C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(\lambda x\,.\,C)[x/B]\|]\,.$$

By definition 42.11 (4), this amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda x\,.\,C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(\lambda x\,.\,C)\|]\,,$$

which is an immediate consequence of λ-abstraction.
5. $A \equiv (\lambda y\,.\,C)$ and $x \not\equiv y$. What has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda y\,.\,C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(\lambda y\,.\,C)[x/B]\|]\,.$$

Distinguish cases according to definition 42.11 in **[15]**, clauses (5) and (6).
5.1. $y \notin FV(B)$ or $y \notin FV(C)$. By definition 42.11 (5) in **[15]**, what has to be shown reduces to

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda y\,.\,C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|\lambda y\,.\,C[x/B]\|]\,.$$

which, by definition 4.1 (3.2) amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \lambda^\dagger yz\, (z = \|C\|)] \Leftrightarrow \mathfrak{F}[\|B\|, \lambda^\dagger yz\, (z = \|C[x/B]\|)]\,,$$

which, in turn, follows by the inductions hypothesis.

5.2. $y \in FV(B)$ and $y \in FV(C)$. By definition 42.11 (6), what has to be shown reduces to

$$\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}^{\mathsf{Z}} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda y \centerdot C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|\lambda z \centerdot C[y/z][x/B]\|]$$

which, by definition 4.1 (3.4) can be reduced to

$$\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}^{\mathsf{Z}} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \lambda^{\dagger} yz\, (z = \|C\|)] \Leftrightarrow \mathfrak{F}[\|B\|, \lambda^{\dagger} zy_1\, (y_1 = \|C[y/z][x/B]\|)]$$

which, in turn, follows by the inductions hypothesis. QED

PROPOSITION 4.6. *Inferences according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-derivable.*

(4.6i)
$$\frac{\langle \|B_2\|, y\rangle \mathrel{\tilde{\in}} \|A_2\|, \Gamma \Rightarrow \langle \|B_1\|, y\rangle \mathrel{\tilde{\in}} \|A_1\|}{\Gamma \Rightarrow \|A_1 B_1\| = \|A_2 B_2\|}$$

where y is a free variable in the upper sequent, which does not occur in the lower sequent.

(4.6ii)
$$\frac{\Gamma \Rightarrow \|B\| = \|A\|}{\langle \|A\|, s\rangle \mathrel{\tilde{\in}} t, \Gamma \Rightarrow \langle \|B\|, s\rangle \mathrel{\tilde{\in}} t}$$

(4.6iii)
$$\frac{\Gamma \Rightarrow \|A[x/x_1]\| = \|B[x/x_1]\|}{\Gamma \Rightarrow \|\lambda x \centerdot A\| = \|\lambda x \centerdot B\|}$$

where $x_1 \notin FV(A)$ and $x_1 \notin FV(B)$.

Proof. Re 4.6i.

$$\begin{array}{c}
\langle \|B_2\|, b\rangle \mathrel{\tilde{\in}} \|A_2\|, \Gamma \Rightarrow \langle \|B_1\|, b\rangle \mathrel{\tilde{\in}} \|A_1\| \qquad\qquad a \in b \Rightarrow a \in b \\
\hline
\langle \|B_1\|, b\rangle \mathrel{\tilde{\in}} \|A_1\| \to a \in b, \langle \|B_2\|, b\rangle \mathrel{\tilde{\in}} \|A_2\|, \Gamma \Rightarrow a \in b \\
\hline
\langle \|B_1\|, b\rangle \mathrel{\tilde{\in}} \|A_1\| \to a \in b, \Gamma \Rightarrow \langle \|B_2\|, b\rangle \mathrel{\tilde{\in}} \|A_2\| \to a \in b \\
\hline\hline
\bigwedge y\, (\langle \|B_1\|, y\rangle \mathrel{\tilde{\in}} \|A_1\| \to a \in y), \Gamma \Rightarrow \bigwedge y\, (\langle \|B_2\|, y\rangle \mathrel{\tilde{\in}} \|A_2\| \to a \in y) \\
\hline\hline
a \in \lambda x \bigwedge y\, (\langle \|B_1\|, y\rangle \mathrel{\tilde{\in}} \|A_1\| \to x \in y), \Gamma \Rightarrow a \in \lambda x \bigwedge y\, (\langle \|B_2\|, y\rangle \mathrel{\tilde{\in}} \|A_2\| \to x \in y) \\
\hline
\Gamma \Rightarrow a \in \|A_1 B_1\| \to a \in \|A_2 B_2\|
\end{array}$$

Analogously for $\varGamma \Rightarrow a\in\|A_2B_2\| \to a\in\|A_1B_1\|$. Finish in a familiar way as follows:

$$\dfrac{\dfrac{\varGamma \Rightarrow a\in\|A_1B_1\| \to a\in\|A_2B_2\| \qquad\qquad \varGamma \Rightarrow a\in\|A_2B_2\| \to a\in\|A_1B_1\|}{\varGamma \Rightarrow a\in\|A_1B_1\| \leftrightarrow a\in\|A_2B_2\|}}{\varGamma \Rightarrow \bigwedge x\,(x\in\|A_1B_1\| \leftrightarrow x\in\|A_2B_2\|)}$$

Re 4.6ii. This is the point where the strength of Z-inferences is needed, and that in the inference marked by † which is according to 135.20vii in [**15**], p. 1847.

$$\begin{array}{c}
\varGamma \Rightarrow \|B\| = \|A\| \\ \hline
\varGamma \Rightarrow \langle\|B\|, s\rangle = \langle\|A\|, s\rangle \\ \hline
\langle\|B\|, s\rangle = b, \varGamma \Rightarrow \langle\|A\|, s\rangle = b \qquad\qquad b\in t \Rightarrow b\in t \\ \hline
\langle\|A\|, s\rangle = b \supset b\in t, \varGamma \Rightarrow \langle\|B\|, s\rangle = b \supset b\in t \quad \dagger \\ \hline
\bigwedge y\,(\langle\|A\|, s\rangle = y \supset y\in t), \varGamma \Rightarrow \langle\|B\|, s\rangle = b \supset b\in t \\ \hline
\bigwedge y\,(\langle\|A\|, s\rangle = y \supset y\in t), \varGamma \Rightarrow \bigwedge y\,(\langle\|B\|, s\rangle = y \supset y\in t) \\ \hline\hline
\langle\|A\|, s\rangle \tilde{\in} t, \varGamma \Rightarrow \langle\|B\|, s\rangle \tilde{\in} t
\end{array}$$

Re 4.6iii.

$$\begin{array}{c}
\dfrac{c = \langle\|x_1\|, b\rangle \Rightarrow c = \langle\|x_1\|, b\rangle \qquad \begin{array}{c} \dfrac{\varGamma \Rightarrow \|A[x/x_1]\| = \|B[x/x_1]\|}{b = \|A[x/x_1]\|, \varGamma \Rightarrow b = \|B[x/x_1]\|} \\ \hline\hline \|x_1\|\in\lambda x\,(b = \|A\|), \varGamma \Rightarrow \|x_1\|\in\lambda x\,(b = \|B\|) \end{array} \;4.5}{c = \langle\|x_1\|, b\rangle \supset \|x_1\|\in\lambda x\,(b = \|A\|) \Rightarrow c = \langle\|x_1\|, b\rangle \supset \|x_1\|\in\lambda x\,(b = \|B\|)} \\ \hline\hline
\bigwedge x_1 \bigwedge y(c = \langle x_1, y\rangle \supset x_1\in\lambda x\,(y = \|A\|)) \Rightarrow c = \langle\|x_1\|, b\rangle \supset \|x_1\|\in\lambda x\,(b = \|B\|) \\ \hline
c\in\|\lambda x\centerdot A\|, \varGamma \Rightarrow c = \langle\|x_1\|, b\rangle \supset \|x_1\|\in\lambda x\,(b = \|B\|) \\ \hline\hline
c\in\|\lambda x\centerdot A\|, \varGamma \Rightarrow \bigwedge x_1 \bigwedge y(c = \langle x_1, y\rangle \supset x_1\in\lambda x\,(y = \|B\|)) \\ \hline
c\in\|\lambda x\centerdot A\|, \varGamma \Rightarrow c\in\|\lambda x\centerdot B\| \\ \hline
\varGamma \Rightarrow c\in\|\lambda x\centerdot A\| \to c\in\|\lambda x\centerdot B\|
\end{array}$$

Continue as for 4.6i. QED

PROPOSITION 4.7. *If $y_1 \notin FV(B)$ and no variable bound in A is free in B, then there exists a natural number* n *such that*

$$\mathbf{L^iD}_\lambda \vdash \mathrm{n}[\|B\| = \|y_1\|] \Rightarrow \|A[x/B]\| = \|A[x/y_1]\|\,.$$

Proof by induction on the length of A. Distinguish cases according to the form of A. With the exception of the case that $A \equiv (C_1 C_2)$, there is hardly any change to the proof of proposition 4.7 in the TOOLS, so I shall only treat that case.

$A \equiv (C_1 C_2)$. As an immediate consequence of the induction hypothesis and proposition 4.6ii there is a natural number n_1 such that

$$\frac{\mathrm{n}_1[\|B\| = \|y_1\|] \Rightarrow \|C_2[x/B]\| = \|C_2[x/y_1]\|}{\langle \|C_2[x/y_1]\|, b\rangle \tilde{\in} \|C_1[x/y_1]\|, \mathrm{n}_1[\|B\| = \|y_1\|] \Rightarrow \langle \|C_2[x/B]\|, b\rangle \tilde{\in} \|C_1[x/y_1]\|}$$

is $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$-deducible. By the induction hypothesis there is also a natural number n_2 such that

$$\mathbf{L^iD}_{\lambda} \;\vdash\; \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \|C_1[x/y_1]\| = \|C_1[x/B]\|\,.$$

This makes it possible to continue as follows, employing 4.6i:

$$\frac{\langle \|C_2[x/y_1]\|, b\rangle \tilde{\in} \|C_1[x/y_1]\|, \mathrm{n}_1 + \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \langle \|C_2[x/B]\|, b\rangle \tilde{\in} \|C_1[x/B]\|}{\mathrm{n}_1 + \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \|C_1[x/B] C_2[x/B]\| = \|C_1[x/y_1] C_2[x/y_1]\|}$$

By definition 42.11 (3) in [**15**], this is

$$\mathrm{n}_1 + \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \|(C_1 C_2)[x/B]\| = \|(C_1 C_2)[x/y_1]\|\,. \qquad \text{QED}$$

PROPOSITION 4.8. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(4.8i) $\quad \langle \|B\|, b\rangle \tilde{\in} \|\lambda x\,.\,A\| \Rightarrow b = \|A[x/B]\|$

(4.8ii) $\quad s = \|A[x/B]\| \Rightarrow \langle \|B\|, s\rangle \tilde{\in} \|\lambda x\,.\,A\|$

(4.8iii) $\quad \Rightarrow \langle \|B\|, \|A[x/B]\|\rangle \tilde{\in} \|\lambda x\,.\,A\|$

Proof. Re 4.8i.

$$\dfrac{\Rightarrow \langle \|B\|, b\rangle = \langle \|B\|, b\rangle \qquad \dfrac{b = \|A[x/B]\| \Rightarrow b = \|A[x/B]\|}{\|B\| \in \lambda x\,(b = \|A\|) \Rightarrow b = \|A[x/B]\|}}{\dfrac{\langle \|B\|, b\rangle = \langle \|B\|, b\rangle \supset \|B\| \in \lambda x\,(b = \|A\|) \Rightarrow b = \|A[x/B]\|}{\bigwedge x_1 \bigwedge y\,(\langle \|B\|, b\rangle = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}}$$

$$\dfrac{\dfrac{\dfrac{\Rightarrow \langle \|B\|, b\rangle = \langle \|B\|, b\rangle \qquad \langle \|B\|, b\rangle \in \lambda^{\dagger} xy\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}{\langle \|B\|, b\rangle = \langle \|B\|, b\rangle \supset \langle \|B\|, b\rangle \in \lambda^{\dagger} xy\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}}{\bigwedge y_1\,(y_1 = \langle \|B\|, b\rangle \supset y_1 \in \lambda^{\dagger} xy\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}}{\langle \|B\|, b\rangle \tilde{\in} \lambda z \bigwedge x_1 \bigwedge y\,(z = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}$$

Re 4.8ii. Employ 4.7, with z satisfying the necessary requirements.

$$\begin{array}{c}
\mathrm{n}[\|B\| = \|z\|] \Rightarrow \|A[x/B]\| = \|A[x/z]\| \\ \hline
\mathrm{n}[\|B\| = \|z\|], s = \|A[x/B]\| \Rightarrow s = \|A[x/z]\| \\ \hline
\mathrm{n}[\|B\| = \|z\|], s = b, s = \|A[x/B]\| \Rightarrow b = \|A[x/z]\| \\ \hline
\mathrm{n}[\|B\| = \|z\|], s = b, s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\,(b = \|A\|) \\ \hline
\mathrm{n}[\langle \|B\|, s\rangle = \langle \|z\|, b\rangle], s = b, s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\,(b = \|A\|) \\ \hline
\mathrm{n}[\langle \|B\|, s\rangle = \langle \|z\|, b\rangle], \langle \|B\|, s\rangle = \langle \|z\|, b\rangle, s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\,(b = \|A\|) \\ \hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], \mathrm{n}+1[a = \langle \|z\|, b\rangle], s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\,(b = \|A\|) \\ \hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], s = \|A[x/B]\| \Rightarrow a = \langle \|z\|, b\rangle \supset \|z\| \in \lambda x\,(b = \|A\|) \\ \hline\hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], s = \|A[x/B]\| \Rightarrow \bigwedge x_1 \bigwedge y\,(a = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|) \\ \hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], s = \|A[x/B]\| \Rightarrow a \in \lambda^{\dagger} xy\,(y = \|A\|) \\ \hline
s = \|A[x/B]\| \Rightarrow a = \langle \|B\|, s\rangle \supset a \in \lambda^{\dagger} xy\,(y = \|A\|) \\ \hline
s = \|A[x/B]\| \Rightarrow \bigwedge y_1\,(y_1 = \langle \|B\|, s\rangle \supset y_1 \in \lambda^{\dagger} xy\,(y = \|A\|)) \\ \hline
s = \|A[x/B]\| \Rightarrow \langle \|B\|, s\rangle \,\tilde{\in}\, \lambda z \bigwedge x_1 \bigwedge y\,(z = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|))
\end{array}$$

Re 4.8iii. This is an immediate consequence of 4.8ii. QED

REMARK 4.9. 4.6iii, 4.8i and 4.8ii above are the points where the notion of weak implication is really needed, more specifically, a notion of weak implication that satisfies the following schemata

$$\frac{A, \ldots, A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\,, \qquad \frac{A, \Gamma \Rightarrow B}{B \supset C, \Gamma \Rightarrow A \supset C}\,, \qquad \text{and} \qquad \frac{\Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\,.$$

This concludes the listing of the relevant tools. I now begin with the translation of α-conversion.

PROPOSITION 4.10. *If* $y \notin FV(A)$, *then*

$$\mathbf{L^iD}_{\lambda} \vdash \Rightarrow \|\lambda x \,.\, A = \lambda y \,.\, A[x/y]\|\,.$$

Proof. What has to be shown is

$$\mathbf{L^iD}_{\lambda}^{\mathrm{Z}} \vdash \Rightarrow \|\lambda x \,.\, A\| = \|\lambda y \,.\, A[x/y]\|\,.$$

By proposition 4.6, it is sufficient to show

$$\mathbf{L^iD}_{\lambda}^{\mathrm{Z}} \vdash \Rightarrow \|A[x/x_1]\| = \|A[x/y][y/x_1]\|\,,$$

which is obvious in view of proposition 42.14i in [**15**], p. 502, establishing that $A[x/x_1]$ and $A[x/y][y/x_1]$ are actually identical in the sense of definition 42.11 in [**15**], if $y \notin FV(A)$. QED

I continue with the translation of β-conversion.

PROPOSITION 4.11. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(4.11i) $s \in \|(\lambda x\,.\,A)B\| \Rightarrow s \in \|A[x/B]\|$

(4.11ii) $s \in \|A[x/B]\| \Rightarrow s \in \|(\lambda x\,.\,A)B\|$

(4.11iii) $\Rightarrow \|(\lambda x\,.\,A)B = A[x/B]\|$

Proof. Re 4.11i. Employ 4.8iii.

$$\dfrac{\dfrac{\dfrac{\Rightarrow \langle\|B\|, \|A[x/B]\|\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \qquad s \in \|A[x/B]\| \Rightarrow s \in \|A[x/B]\|}{\langle\|B\|, \|A[x/B]\|\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \to s \in \|A[x/B]\| \Rightarrow s \in \|A[x/B]\|}}{\bigwedge y\,(\langle\|B\|, y\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \to s \in y) \Rightarrow s \in \|A[x/B]\|}}{s \in \lambda x_1 \bigwedge y\,(\langle\|B\|, y\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \to x_1 \in y) \Rightarrow s \in \|A[x/B]\|}$$

Re 4.11ii.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s \in \|A[x/B]\| \Rightarrow s \in \|A[x/B]\|}{s \in \|A[x/B]\|, b = \|A[x/B]\| \Rightarrow s \in b}\;4.8\mathrm{i}}{s \in \|A[x/B]\|, \langle\|B\|, b\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\|) \Rightarrow s \in b}}{s \in \|A[x/B]\| \Rightarrow \langle\|B\|, b\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \to s \in b}}{s \in \|A[x/B]\| \Rightarrow \bigwedge y\,(\langle\|B\|, y\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \to s \in y)}}{s \in \|A[x/B]\| \Rightarrow s \in \lambda x_1 \bigwedge y\,(\langle\|B\|, y\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \to x_1 \in y)}$$

Re 4.11iii. This is a straightforward consequence of 4.11i and ii.
Re 4.11iv. This is a straightforward consequence of 4.11iii in view of the definition 4.1 (3.4). QED

PROPOSITION 4.12. *Inferences according to the following schemata are* $\mathbf{L^iD}_{\lambda}$*-derivable.*

(4.12i)
$$\dfrac{\Rightarrow \|A = B\|}{\Rightarrow \|B = A\|}$$

(4.12ii)
$$\frac{\Rightarrow \|A = B\| \qquad \Rightarrow \|B = C\|}{\Rightarrow \|B = C\|}$$

(4.12iii)
$$\frac{\Rightarrow \|A = B\|}{\Rightarrow \|CA = CB\|}$$

Proof. *Re* 4.12i and 4.12ii. These are immediate consequences of the way $=$ is defined in $\mathbf{L^iD}_\lambda$.
Re 4.12iii. This is a consequence of the inclusive character built into the definition of $\tilde{\in}$.

$$\cfrac{\cfrac{\Rightarrow \|A\| = \|B\|}{\langle \|B\|, y\rangle \,\tilde{\in}\, \|C\| \Rightarrow \langle \|A\|, y\rangle \,\tilde{\in}\, \|C\|}\ 4.6\text{ii}}{\Rightarrow \|CA\| = \|CB\|}\ 4.6\text{i}$$

QED

THEOREM 4.13. *If* $\lambda\beta \vdash A$, *then* $\mathbf{L^iD}^{\mathrm{Z}}_\lambda \vdash \Rightarrow \|A\|$.

Proof. 4.12i-iii are the translations of (σ), (τ) and (μ), respectively. QED

THEOREM 4.14. *Not every equation is* $\lambda\beta$*-deducible.*

Proof. This is an immediate consequence of the foregoing theorem 4.13 and the fact that $\|x = y\|$, *i.e.*, $a = b$, ist not an $\mathbf{L^iD}^{\mathrm{Z}}_\lambda$-deducible wff. QED

Discussion. In view of the smooth interpretation of logic in illative combinatory logic provided in [**11**],[13] the question will arise why is the translation provided here rather awkward in comparison? My answer is to draw attention to the notion of equality. The notion of equality in $\mathbf{L^iD}_\lambda$ (and, of course, $\mathbf{L^iD}^{\mathrm{Z}}_\lambda$) is provided by implication, conjunction, generalization (*i.e.*, illative notions) and elementhood in the usual way:

$$s = t :\equiv \bigwedge x\, ((x \in s \to x \in t) \wedge (x \in t \to x \in s))\,.$$

This, however, does not agree too well with the introduction of $=$ as a primitive relation in λ-calculus and combinatory logic. Consider the

[13] See, in particular, p. 587.

following situation:

$$\cfrac{CA \qquad \cfrac{\cfrac{A \to B \qquad B \to A}{A = B}}{CA = CB}\,(\mu)}{CB}\,Eq$$

which has already been shown to be incompatible in remark 42.64 (2) in [**15**], p. 517.

This doesn't seem to be too surprising if one considers the definition of $B \in A$ in [**11**], p. 587, as AB. The μ-inference together with the Eq-inference then reads:

$$\frac{A \in C \qquad A = B}{B \in C}$$

which just displays the characteristic feature of extensionality from a set theoretical perspective.

In other words, given the reading of equality in $\mathbf{L^iD^Z_\lambda}$, (μ) does actually provide a form of weak extensionality as considered in [**11**], p. 594, which has been shown to be incompatible with a formalized theory equivalent to $BCK\lambda\beta$ in U. [**15**], p. 517.

This, however, is compensated in the awkward definition of AB in the LD-translation as $\lambda x \bigwedge y\, (\langle B, y\rangle \tilde{\in} A \to x \in y)$ by the somewhat "inclusive" notion $\tilde{\in}$.

Differently put: the system $BCK\lambda\beta$ from [**11**] becomes trivial, if something like

$$\frac{A \to B \qquad B \to A}{A = B}$$

is added as a basic rule of deduction (i.e., the premisses not depending on open assumptions).

5. Addition 137f. An approach to extending $\mathbf{L^iD^Z_\lambda}$ to accomodate nested double induction and recursion

This addition is still more of a suggestion than a fully worked out approach. The reason that it is included here is that it gives the idea of how I want to extend the approach begun with my **Z**-inferences to gain more deductive strength in systems of higher order logic without contraction.

5a. Introduction. Primitive recursion (or 1-recursion) is available in $\mathbf{L^iD^Z_\lambda}$ (in the sense that functions defined by primitive recursion from total functions can be explicitly defined and proven to be total), as shown in [**16**], but not so k-recursion for $k > 1$. The latter is readily concluded from a simple ordinal observation: a consistency proof for 2-recursion requires an induction up to ω^{ω^ω}, while that of $\mathbf{L^iD^Z_\lambda}$ can be shown by an induction up to ω^ω. On the other hand, as I suggested at the end of [**16**], given a certain reinforced necessity operator obeying the rules

$$\frac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{\boxed{\Box} A, \Gamma \Rightarrow C} \qquad \text{and} \qquad \frac{\Box^{\mathrm{n}} A \Rightarrow C}{\boxed{\Box} A \Rightarrow \boxed{\Box} C}\ ,$$

there is an easy way to overcome the difficulties. The present paper is dedicated to a way of introducing such a reinforced necessity operator $\boxed{\Box}$ *without* adding any new primitive symbols.[14]

Remark 5.1. Regarding the reduction step for the above rules: if the last part of a deduction has the form

$$\dfrac{\dfrac{\Box^{\mathrm{m}} B \Rightarrow A}{\boxed{\Box} B \Rightarrow \boxed{\Box} A} \qquad \dfrac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{\boxed{\Box} A, \Gamma \Rightarrow C}}{\boxed{\Box} B, \Gamma \Rightarrow C}\ \clubsuit\ ,$$

then a deduction can be constructed as follow:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Box^{\mathrm{m}} B \Rightarrow A}}{\overline{\Box^{\mathrm{m+n}} B \Rightarrow \Box^{\mathrm{n}} A}} \qquad \dfrac{\vdots}{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}}{\dfrac{\Box^{\mathrm{m+n}} B, \Gamma \Rightarrow C}{\boxed{\Box} B, \Gamma \Rightarrow C}}\ \clubsuit\ .$$

This may look pretty innocent. But since it is sufficient to provide 2-recursion, it will come no cheaper than by an induction up to ω^{ω^ω}.

---

[14] No relief is to be expected from the introduction of function variables as promoted in, *e.g.*, [**9**] and [**8**] for the formulation of k-recursion for $k > 1$, simply because the problem in the present approach is not the *formulation* of an appropriate term, but the nested double induction required in the proof that it satisfies the criterion of a function: uniqueness of the value; and that problem prevails.

What I need is a way of quantifying, as it were, over necessity operators, and that in a way that allows a form of induction similar to that provided by $\breve{\mathbf{\Pi}}^{\circ}$ in [**14**]. This is what I am going to provide now.

My approach to providing sufficient deductive strength for proving 2-recursion is based heavily on [**14**] and [**16**] and is a further extension of the system $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$ presented in [**14**].

5b. $\mathbf{\Psi}_2$ and $\mathbf{Z}_2$. I begin by introducing a new kind of successor notion.

DEFINITIONS 5.2. (1) $s^{2I} :\equiv \lambda x\, \Box(x \in s)$ ("necessor", a kind of successor with regard to the necessity operator, a nec[essity-succ]essor).
(2) The set $\mathbf{\Psi}_2$ is defined inductively as follows:

(2.1) I is an element of $\mathbf{\Psi}_2$;
(2.1) If t is an element $\mathbf{\Psi}_2$, then so is t^{2I}.

(3) If n is a natural number, then the *corresponding* $\mathbf{\Psi}_2$-*element* is defined inductively as follows:

(3.1) I is the *corresponding* $\mathbf{\Psi}_2$-*element* to 0;
(3.2) If $\bar{\bar{n}}$ is the corresponding $\mathbf{\Psi}_2$-element to n, then $\bar{\bar{n}}^{2I}$ is the *corresponding* $\mathbf{\Psi}_2$-element to n′.

EXAMPLE 5.3. $I^{2I2I} \equiv \lambda x_1\, \Box(x_1 \in \lambda x_2\, \Box(x_2 \in I)) = \lambda x\, \Box\Box(x \in I)$.

CONVENTION 5.4. I shall use $\bar{\bar{m}}$ and $\bar{\bar{n}}$ as syntactic symbols for elements of $\mathbf{\Psi}_2$, possibly with index numbers.

REMARK 5.5. $\mathbf{\Psi}_2$ is the set $\{I, I^{2I}, I^{2I2I}, \ldots\}$, where

$I^{2I} = \lambda x\, \Box(x = \mathcal{V})$,
$I^{2I2I} = \lambda x\, \Box\Box(x = \mathcal{V})$,
. . . .

Note, however, that this is equality, not identity! What is aimed at is, of course, this: $[A/\bar{\bar{n}}] \leftrightarrow \Box^{\mathrm{n}} A$.

PROPOSITION 5.6. *If $s \in \mathbf{\Psi}_2$, then there exists a natural number* n *such that s is the corresponding element of* n *in* $\mathbf{\Psi}_2$.

Proof. As for the case of $\mathbf{\Psi}$ in proposition 131.9 in [**15**], p. 1789, this is an immediate consequence of the definition of corresponding element. QED

PROPOSITION 5.7. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$-*deducible.*

(5.7i) $[A/s^{2I}] \Leftrightarrow \Box[A/s]$

(5.7ii) $\Box([A/s] \mathbin{\square} [B/s] \to [A \mathbin{\square} B/s]) \Rightarrow [A/s^{2I}] \mathbin{\square} [B/s^{2I}] \to [A \mathbin{\square} B/s^{2I}]$

(5.7iii) $[A/s^{2I}] \Rightarrow [A/s]$

(5.7iv) $[A/I^{2I}] \Leftrightarrow \Box A$

Proof. *Re* 5.7i. Immediate consequence of the abstraction rules.
Re 5.7ii.

$$\begin{array}{c}
[A/s] \mathbin{\square} [B/s] \to [A \mathbin{\square} B/s], [A/s], [B/s] \Rightarrow [A \mathbin{\square} B/s] \\ \hline
\Box([A/s] \mathbin{\square} [B/s] \to [A \mathbin{\square} B/s]), \Box[A/s], \Box[B/s] \Rightarrow \Box[A \mathbin{\square} B/s] \\ \hline\hline
\Box([A/s] \mathbin{\square} [B/s] \to [A \mathbin{\square} B/s]), [A/s^{2I}], [B/s^{2I}] \Rightarrow [A \mathbin{\square} B/s^{2I}] \\ \hline
\Box([A/s] \mathbin{\square} [B/s] \to [A \mathbin{\square} B/s]), [A/s^{2I}] \mathbin{\square} [B/s^{2I}] \Rightarrow [A \mathbin{\square} B/s^{2I}] \\ \hline
\Box([A/s] \mathbin{\square} [B/s] \to [A \mathbin{\square} B/s]) \Rightarrow [A/s^{2I}] \mathbin{\square} [B/s^{2I}] \to [A \mathbin{\square} B/s^{2I}]
\end{array}$$

Re 5.7iii.

$$\begin{array}{c}
\lambda A \in s \Rightarrow [A/s] \\ \hline
\Box(\lambda A \in s) \Rightarrow [A/s] \\ \hline
[A/s^{2I}] \Rightarrow [A/s]
\end{array}$$

Re 5.7iv. This is a straightforward consequence of 5.7i and 131.15i and ii in [**15**], p. 1792. QED

As for the case of $\mathbf{\Psi}$, the problem consists in capturing the informal notion $\mathbf{\Psi}_2$ on the formal level, and that in a way which provides for a form of induction. As in the case of $\mathbf{Z}$, the point is to find an application of self-reference (fixed-point construction) which creates, as it were, its own "successor", this time with regard to the necessity operator, *i.e.*, its own *necessor*. This is what the following definition aims at.

DEFINITION 5.8. $\breve{\gamma}_2[A] :\equiv \lambda x \, \Box(x \in x \mathbin{\square} A) \in \lambda x \, \Box(x \in x \mathbin{\square} A)$.

The next proposition lists a number of properties of $\breve{\gamma}_2$, somewhat paralleling proposition 132.5 in [**15**], p. 1804, concerning the case of $\breve{\gamma}$.

PROPOSITION 5.9. *Sequents according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.9i) $\breve{\gamma}_2[A] \Leftrightarrow \Box(\breve{\gamma}_2[A] \mathbin{\scriptstyle\square} A)$

(5.9ii) $\breve{\gamma}_2[A] \Rightarrow \Box A$

(5.9iii) $\breve{\gamma}_2[A] \Rightarrow A$

(5.9iv) $\breve{\gamma}_2[A] \Rightarrow \Box^{\mathrm{n}} A$

(5.9v) $\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A]$

(5.9vi) $\breve{\gamma}_2[A] \Rightarrow \Box^{\mathrm{n}}\breve{\gamma}_2[A]$

(5.9vii) $\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A] \mathbin{\scriptstyle\square} \Box A$

(5.9viii) $\breve{\gamma}_2[A \wedge B] \Rightarrow \breve{\gamma}_2[A \wedge B] \mathbin{\scriptstyle\square} \Box^{\mathrm{n}} A$

(5.9ix) $\breve{\gamma}_2[A \wedge B] \Rightarrow \breve{\gamma}_2[A \wedge B] \mathbin{\scriptstyle\square} \Box^{\mathrm{n}} B$

Proof. *Re* 5.9i. Straightforward consequence of the abstraction rules.
Re 5.9ii.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\breve{\gamma}_2[A], A \Rightarrow A}}{\breve{\gamma}_2[A] \mathbin{\scriptstyle\square} A \Rightarrow A}}{\Box(\breve{\gamma}_2[A] \mathbin{\scriptstyle\square} A) \Rightarrow \Box A}}{\breve{\gamma}_2[A] \Rightarrow \Box A}$$

Re 5.9iii. Immediate consequence of 5.9ii.
Re 5.9iv. Repeat 5.9i.
Re 5.9v.

$$\dfrac{\dfrac{\dfrac{\dfrac{\breve{\gamma}_2[A] \Rightarrow \breve{\gamma}_2[A]}{\breve{\gamma}_2[A], A \Rightarrow \breve{\gamma}_2[A]}}{\breve{\gamma}_2[A] \mathbin{\scriptstyle\square} A \Rightarrow \breve{\gamma}_2[A]}}{\Box(\breve{\gamma}_2[A] \mathbin{\scriptstyle\square} A) \Rightarrow \Box\breve{\gamma}_2[A]}}{\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A]}$$

Re 5.9vi. Employ an induction on n, approaching along the line of 5.9v.

Re 5.9vii. Employ 5.9vi and 5.9ii:

$$\dfrac{\breve{\gamma}_2[A]\Rightarrow\Box\Box\breve{\gamma}_2[A] \qquad \dfrac{\dfrac{\Box\breve{\gamma}_2[A]\Rightarrow\Box\breve{\gamma}_2[A] \qquad \dfrac{\breve{\gamma}_2[A]\Rightarrow\Box A}{\Box\breve{\gamma}_2[A]\Rightarrow\Box A}}{\Box\breve{\gamma}_2[A],\Box\breve{\gamma}_2[A]\Rightarrow\Box\breve{\gamma}_2[A]\mathbin{\scriptstyle\Box}\Box A}}{\Box\Box\breve{\gamma}_2[A]\Rightarrow\Box\breve{\gamma}_2[A]\mathbin{\scriptstyle\Box}\Box A}}{\breve{\gamma}_2[A]\Rightarrow\Box\breve{\gamma}_2[A]\mathbin{\scriptstyle\Box}\Box A}\,\clubsuit$$

Re 5.9viii and 5.9ix. These are straightforward consequences of the foregoing results. QED

COROLLARY 5.10. *Inferences according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-derivable.*

(5.10i) $$\dfrac{A,\mathit{\Gamma}\Rightarrow C}{\breve{\gamma}_2[A],\mathit{\Gamma}\Rightarrow C}$$

(5.10ii) $$\dfrac{\Box A,\mathit{\Gamma}\Rightarrow C}{\breve{\gamma}_2[A],\mathit{\Gamma}\Rightarrow C}$$

(5.10iii) $$\dfrac{\Box^{\mathrm{n}}A,\mathit{\Gamma}\Rightarrow C}{\breve{\gamma}_2[A],\mathit{\Gamma}\Rightarrow C}$$

(5.10iv) $$\dfrac{\breve{\gamma}_2[A\wedge B],\Box^{\mathrm{n}}A,\mathit{\Gamma}\Rightarrow C}{\breve{\gamma}_2[A\wedge B],\mathit{\Gamma}\Rightarrow C}$$

(5.10v) $$\dfrac{\breve{\gamma}_2[A\wedge B],\Box^{\mathrm{n}}B,\mathit{\Gamma}\Rightarrow C}{\breve{\gamma}_2[A\wedge B],\mathit{\Gamma}\Rightarrow C}$$

(5.10vi) $$\dfrac{\Box^{\mathrm{n}}\breve{\gamma}_2[A],\mathit{\Gamma}\Rightarrow C}{\breve{\gamma}_2[A],\mathit{\Gamma}\Rightarrow C}$$

Next comes the definition of a term that is meant to do for the necessor ^{2I} what $\mathbf{Z}$ did for the verisection ^{I}.[15]

DEFINITION 5.11.
$\mathbf{Z}_2 :\equiv \lambda x \bigwedge y\,(\breve{\gamma}_2[I\in y\wedge \bigwedge z\,(\Box(z\in y)\rightarrow z^{2I}\in y)]\rightarrow x\in y)\,.$

[15] As regards the term "verisection", cf. definition 131.5 on p. 1788 of [**15**].

REMARK 5.12. This definition of $\mathbf{Z}_2$ is designed with an eye to a possible consistency proof somewhat along similar lines as that sketched in §133 of [**15**] for the case of $\mathbf{L^iD}_\lambda^{\mathrm{Z}}$. At first sight, it may look as if the approach from [**14**] could be easily adapted from ^{I} to ^{2I}. This however, runs into trouble at the following point: while

$$\frac{\Rightarrow A}{\lambda A \in s \Rightarrow \lambda A \in s^I}$$

is perfectly $\mathbf{L^iD}_\lambda$-deducible, the following isn't $\mathbf{L^iD}_\lambda^{\mathrm{Z}}$-deducible:

$$\frac{\Rightarrow A}{\lambda A \in s \Rightarrow \Box(\lambda A \in s)} \; .$$

A similar consideration applies to the employment of a fixed-point à la [**18**], *i.e.*, $\mathbf{Z}_2 = \lambda x \bigwedge y\,(I \in y \,\square\, \bigwedge z\,(z \in \mathbf{Z}_2 \to z^{2I} \in y) \to x \in y)$. It is with regard to this problem that the necessity operator is introduced in front of the "induction hypothesis", *i.e.*, the sub-formula $(z \in y)$ in $\mathbf{Z}_2$.

PROPOSITION 5.13. *Sequents according to the following schemata are* $\mathbf{L^iD}_\lambda^{\mathrm{Z}}$*-deducible.*

(5.13i) $\Rightarrow I \in \mathbf{Z}_2$

(5.13ii) $\Box(s \in \mathbf{Z}_2) \Rightarrow s^{2I} \in \mathbf{Z}_2$

Proof. Re 5.13i. Employ 5.9iii:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\breve{\gamma}_2[I \in b] \Rightarrow I \in b}{\breve{\gamma}_2[I \in b], \breve{\gamma}_2[\bigwedge z\,(\Box(z \in b) \to z^{2I} \in b)] \Rightarrow I \in b}}{\breve{\gamma}_2[I \in b] \,\square\, \breve{\gamma}_2[\bigwedge z\,(\Box(z \in b) \to z^{2I} \in b)] \Rightarrow I \in b}}{\Rightarrow \breve{\gamma}_2[I \in b] \,\square\, \breve{\gamma}_2[\bigwedge z\,(\Box(z \in b) \to z^{2I} \in b)] \to I \in b}}{\Rightarrow \bigwedge y\,(\breve{\gamma}_2[I \in y] \,\square\, \breve{\gamma}_2[\bigwedge z\,(\Box(z \in y) \to z^{2I} \in y)] \to I \in y)}}{\Rightarrow I \in \lambda x \bigwedge y\,(\breve{\gamma}_2[I \in y] \,\square\, \breve{\gamma}_2[\bigwedge z\,(\Box(z \in y) \to z^{2I} \in y)] \to x \in y)}$$

Re 5.13ii. Let $\mathfrak{Zet}_2 :\equiv \breve{\gamma}_2[I\in *_1 \wedge \bigwedge z\,(\Box(z\in *_1) \to z^{2I}\in *_1)]$:

$$\begin{array}{c}
\begin{array}{c}
\mathfrak{Zet}_2[b] \Rightarrow \mathfrak{Zet}_2[b] \qquad s\in b \Rightarrow s\in b \\ \hline
\mathfrak{Zet}_2[b] \to s\in b, \mathfrak{Zet}_2[b] \Rightarrow s\in b \\ \hline
\bigwedge y\,(\mathfrak{Zet}_2[y] \to s\in y), \mathfrak{Zet}_2[b] \Rightarrow s\in b \\ \hline
s\in\lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to s\in y), \mathfrak{Zet}_2[b] \Rightarrow s\in b \\ \hline
\Box(s\in\lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to s\in y)), \Box\mathfrak{Zet}_2[b] \Rightarrow \Box(s\in b) \\ \hline
\Box(s\in\lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to s\in y)), \mathfrak{Zet}_2[b] \Rightarrow \Box(s\in b)
\end{array}\ \text{5.10vi} \qquad s^{2I}\in b \Rightarrow s^{2I}\in b \\ \hline
\Box(s\in\mathbf{Z}_2), \mathfrak{Zet}_2[b], \Box(s\in b) \to s^{2I}\in b \Rightarrow s^{2I}\in b \\ \hline
\Box(s\in\mathbf{Z}_2), \mathfrak{Zet}_2[b], \bigwedge z\,(\Box(z\in b) \to z^{2I}\in b) \Rightarrow s^{2I}\in b \\ \hline
\Box(s\in\mathbf{Z}_2), \mathfrak{Zet}_2[b] \Rightarrow s^{2I}\in b \quad \text{(5.10v)} \\ \hline
\Box(s\in\mathbf{Z}_2) \Rightarrow \mathfrak{Zet}_2[b] \to s^{2I}\in b \\ \hline
\Box(s\in\mathbf{Z}_2) \Rightarrow \bigwedge y\,(\mathfrak{Zet}_2[y] \to s^{2I}\in y) \\ \hline
\Box(s\in\mathbf{Z}_2) \Rightarrow s^{2I}\in\lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to x\in y)
\end{array}$$

QED

REMARK 5.14. Notice that $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$ is indeed required in the above deduction of 5.13ii.

5c. $\mathbf{Z}_2$-inferences and $\breve{\mathbf{\Pi}}^{\circ}_2$. As in the case of $\mathbf{Z}$, I shall next proceed to defining terms which provide for some form of proto-induction, in the present case a nested double one.

DEFINITIONS 5.15. (1) An inference according to the following schema is called a $\mathbf{Z}_2$-inference:

$$\frac{\Gamma \Rightarrow s\in\mathbf{Z}_2 \qquad \Rightarrow A}{\Gamma \Rightarrow \lambda A\in s}\ .$$

(2) The formalized theory $\mathbf{L^iD}^{\mathrm{Z}_2}_{\lambda}$ is defined as $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$ plus all $\mathbf{Z}_2$-inferences.

Next comes the definition of $\breve{\mathbf{\Pi}}^{\circ}_2$. In what follows, I shall commonly write $[A/s]$ for $\lambda A \in s$, as I already did in [**14**].

DEFINITIONS 5.16. (1) $\mathfrak{Pi}_2[s,t] :\equiv [I \in t/s] \wedge \bigwedge z\, [\Box(z \in t) \to z^{2I} \in t/s]$.
(2) $\breve{\mathbf{\Pi}}^{\circ}_2 :\equiv \lambda x\, (\Box(x \in \mathbf{Z}_2) \mathbin{\scriptstyle\square} \bigwedge y\, (\mathfrak{Pi}_2[x,y] \supset x \in y))$.

PROPOSITION 5.17. *Inferences according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

(5.17i)
$$\frac{\Rightarrow \mathfrak{F}[I] \qquad \Box\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{2I}]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}$$

(5.17ii)
$$\frac{A \Rightarrow B}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s]}$$

(5.17iii)
$$\frac{\Box A \Rightarrow B}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box[A/s] \Rightarrow [B/s]}$$

(5.17iv)
$$\frac{\Gamma \Rightarrow B}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow [B/s]}$$

Proof. Re 5.17i. Let $\xi :\equiv \lambda x\, \mathfrak{F}[x]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \mathfrak{F}[I]}{\Rightarrow I \in \lambda x\, \mathfrak{F}[x]}}{s \in \mathbf{Z}_2 \Rightarrow [I \in \lambda x\, \mathfrak{F}[x]/s]} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Box\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{2I}]}{\Box(a \in \lambda x\, \mathfrak{F}[x]) \Rightarrow a^{2I} \in \lambda x\, \mathfrak{F}[x]}}{\Rightarrow \Box(a \in \lambda x\, \mathfrak{F}[x]) \to a^{2I} \in \lambda x\, \mathfrak{F}[x]}}{s \in \mathbf{Z}_2 \Rightarrow [\Box(a \in \xi) \to a^{2I} \in \xi/s]}}{s \in \mathbf{Z}_2 \Rightarrow \bigwedge z\, [\Box(z \in \xi) \to z^{2I} \in \xi/s]}}{s \in \mathbf{Z}_2 \Rightarrow [I \in \xi/s] \wedge \bigwedge z\, [\Box(z \in \xi) \to z^{2I} \in \xi/s]} \quad \dfrac{\mathfrak{F}[s] \Rightarrow \mathfrak{F}[s]}{s \in \lambda x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[s]}}{\dfrac{\Box(s \in \mathbf{Z}_2), [I \in \xi/s] \wedge \bigwedge z\, [\Box(z \in \xi) \to z^{2I} \in \xi/s] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}{\dfrac{\Box(s \in \mathbf{Z}_2), \bigwedge y\, ([I \in y/s] \wedge \bigwedge z\, [\Box(z \in y) \to z^{2I} \in y/s] \supset s \in y) \Rightarrow \mathfrak{F}[s]}{\dfrac{\Box(s \in \mathbf{Z}_2) \mathbin{\scriptstyle\square} \bigwedge y\, ([I \in y/s] \wedge \bigwedge z\, [\Box(z \in t) \to z^{2I} \in y/s] \supset s \in y) \Rightarrow \mathfrak{F}[s]}{s \in \lambda x\, (\Box(s \in \mathbf{Z}_2) \mathbin{\scriptstyle\square} \bigwedge y\, ([I \in y/s] \wedge \bigwedge z\, [\Box(z \in t) \to z^{2I} \in y/s] \supset s \in y)) \Rightarrow \mathfrak{F}[s]}}}}$$

Re 5.17ii. Similar to 134.10ii in [**15**], p. 1830:

$$\dfrac{\begin{array}{c} A \Rightarrow B \\ \hline\hline [A/I] \Rightarrow [B/I] \\ \hline \Rightarrow [A/I] \rightarrow [B/I] \end{array} \qquad \begin{array}{c} [A/c] \Rightarrow [A/c] \qquad [B/c] \Rightarrow [B/c] \\ \hline [A/c] \rightarrow [B/c], [A/c] \Rightarrow [B/c] \\ \hline \Box([A/c] \rightarrow [B/c]), \Box[A/c] \Rightarrow \Box([B/c]) \\ \hline\hline \Box([A/c] \rightarrow [B/c]), [A/c^{2I}] \Rightarrow [B/c^{2I}] \\ \hline \Box([A/c] \rightarrow [B/c]) \Rightarrow [A/c^{2I}] \rightarrow [B/c^{2I}] \end{array}}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s]}\ 5.17\mathrm{i}$$

Re 5.17iii.

$$\dfrac{\begin{array}{c} \Box[A/I] \Rightarrow \Box A \quad \Box A \Rightarrow B \\ \hline \Box[A/I \Rightarrow B \\ \hline \Box[A/I] \Rightarrow [B/I] \\ \hline \Rightarrow \Box[A/I \rightarrow [B/I] \end{array} \clubsuit \qquad \begin{array}{c} \Box[A/a] \Rightarrow \Box[A/a] \qquad [B/a] \Rightarrow [B/a] \\ \hline \Box[A/a] \rightarrow [B/a], \Box[A/a] \Rightarrow [B/a] \\ \hline \Box[A/a] \rightarrow [B/a], [A/a^{2I}] \Rightarrow [B/a] \\ \hline\hline \Box(\Box[A/a] \rightarrow [B/a], \Box[A/a^{2I}] \Rightarrow \Box[B/a] \\ \hline \Box(\Box[A/a] \rightarrow [B/a], \Box[A/a^{2I}] \Rightarrow [B/a^{2I}] \\ \hline \Box(\Box[A/a] \rightarrow [B/a]) \Rightarrow \Box[A/a^{2I}] \rightarrow [B/a^{2I}] \end{array}}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box[A/s] \Rightarrow [B/s]}\ 5.17\mathrm{i}.$$

Re 5.17iv. This is just a generalization of 5.17ii by taking the finite box-conjunction of the wffs of $\mathit{\Gamma}$. Left to the reader. QED

REMARK 5.18. There is something bordering on triviality in the "induction steps" of the proofs of 5.17ii–5.17iv, which is essentially due to the $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$-deducibility of the sequent $\Box(\lambda A \in a) \Rightarrow \lambda A \in a^{2I}$:

$$\dfrac{\Box(\lambda A \in a) \Rightarrow \Box(\lambda A \in a)}{\Box(\lambda A \in a) \Rightarrow \lambda A \in a^{2I}}\ .$$

I suggest that this be seen in the context of the difference between a complete induction and a transinite induction. Every ordinal below ω can actually be reached by starting from 0 and adding 1, whereas with a transfinite ordinal one can only say that 0 can be reached by every descending chain.

PROPOSITION 5.19. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.19i) $s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\mathbf{Z}_2)$

(5.19ii) $s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathfrak{I}}\in\mathbf{Z}_2$

(5.19iii) $s\in\breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow A$

(5.19iv) $\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_2$

(5.19v) $s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{Pi}_2[s^{\mathfrak{I}}, t] \Rightarrow \mathfrak{Pi}_2[s, t^{\mathfrak{I}}]$

(5.19vi) $s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box(s\in t), \mathfrak{Pi}_2[s^{\mathfrak{I}}, t] \Rightarrow s^{\mathfrak{I}}\in t$

(5.19vii) $[\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2} \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2$

(5.19viii) $s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)$

(5.19ix) $s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\Box\, s\in\breve{\mathbf{\Pi}}^{\circ}_2$

(5.19x) $s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2$

Proof. Re 5.19i. Fairly immediate consequence of the definition; left to the reader.
Re 5.19ii. This is a straightforward consequence of 5.13ii and 5.19i.
Re 5.19iii.

$$\dfrac{\dfrac{\dfrac{A\Rightarrow A}{[A/I]\Rightarrow A}}{\Rightarrow [A/I]\to A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box(\lambda A\in c)\Rightarrow\Box[A/c]}{[A/c^{\mathfrak{I}}]\Rightarrow\Box[A/c]} \qquad \Box A\Rightarrow A}{\Box[A/c]\to\Box A, [A/c^{\mathfrak{I}}]\Rightarrow A}}{\Box([A/c]\to A), [A/c^{\mathfrak{I}}]\Rightarrow A}}{\Box([A/c]\to A)\Rightarrow [A/c^{\mathfrak{I}}]\to A}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, [A/s]\Rightarrow A}\ 5.17\mathrm{i}\,.$$

Re 5.19iv. Obvious.

Re 5.19v. Inferences according to 5.17iii are marked by $*$:

Left branch:

$$
\begin{array}{c}
\Box(I\in t) \Rightarrow \Box(I\in t) \\ \hline
\Box(I\in t) \Rightarrow I\in t^{\mathfrak{A}} \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box[I\in t/s] \Rightarrow [I\in t^{\mathfrak{A}}/s] \quad * \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, [I\in t/s^{\mathfrak{A}}] \Rightarrow [I\in t^{\mathfrak{A}}/s] \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{Pi}_2[s^{\mathfrak{A}}, t] \Rightarrow [I\in t^{\mathfrak{A}}/s]
\end{array}
$$

Right branch:

$$
\begin{array}{c}
\begin{array}{c}\Box(c\in t) \Rightarrow \Box(c\in t) \\ \hline \Box(c\in t) \Rightarrow c\in t^{\mathfrak{A}}\end{array} \qquad c^{\mathfrak{A}}\in t \Rightarrow c^{\mathfrak{A}}\in t \\ \hline
\Box(c\in t) \to c^{\mathfrak{A}}\in t, c\in t^{\mathfrak{A}} \Rightarrow c^{\mathfrak{A}}\in t \\ \hline
\Box(\Box(c\in t) \to c^{\mathfrak{A}}\in t), \Box(c\in t^{\mathfrak{A}}) \Rightarrow \Box(c^{\mathfrak{A}}\in t) \\ \hline
\Box(\Box(c\in t) \to c^{\mathfrak{A}}\in t), \Box(c\in t^{\mathfrak{A}}) \Rightarrow c^{\mathfrak{A}}\in t^{\mathfrak{A}} \\ \hline
\Box(\Box(c\in t) \to c^{\mathfrak{A}}\in t) \Rightarrow \Box(c\in t^{\mathfrak{A}}) \to c^{\mathfrak{A}}\in t^{\mathfrak{A}} \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box[\Box(c\in t) \to c^{\mathfrak{A}}\in t/s] \Rightarrow [\Box(c\in t^{\mathfrak{A}}) \to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s] \quad * \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, [\Box(c\in t) \to c^{\mathfrak{A}}\in t/s^{\mathfrak{A}}] \Rightarrow [\Box(c\in t^{\mathfrak{A}}) \to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s] \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \bigwedge z[\Box(z\in t) \to z^{\mathfrak{A}}\in t/s^{\mathfrak{A}}] \Rightarrow [\Box(c\in t^{\mathfrak{A}}) \to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s] \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{Pi}_2[s^{\mathfrak{A}}, t] \Rightarrow [\Box(c\in t^{\mathfrak{A}}) \to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s] \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{Pi}_2[s^{\mathfrak{A}}, t] \Rightarrow \bigwedge z\,[\Box(z\in t^{\mathfrak{A}}) \to z^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]
\end{array}
$$

Conclusion from both branches:

$$
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{Pi}_2[s^{\mathfrak{A}}, t] \Rightarrow [I\in t^{\mathfrak{A}}/s] \wedge \bigwedge z\,[\Box(z\in t^{\mathfrak{A}}) \to z^{\mathfrak{A}}\in t^{\mathfrak{A}}/s] .
$$

Re 5.19vi. Employ 5.19iii:

$$
\begin{array}{c}
\Box(s\in t) \Rightarrow \Box(s\in t) \qquad s^{\mathfrak{A}}\in t \Rightarrow s^{\mathfrak{A}}\in t \\ \hline
\Box(s\in t), \Box(s\in t) \to s^{\mathfrak{A}}\in t \Rightarrow s^{\mathfrak{A}}\in t \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box(s\in t), [\Box(s\in t) \to s^{\mathfrak{A}}\in t/s] \Rightarrow s^{\mathfrak{A}}\in t \quad \text{5.19iii} \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box(s\in t), [\Box(s\in t) \to s^{\mathfrak{A}}\in t/s^{\mathfrak{A}}] \Rightarrow s^{\mathfrak{A}}\in t \quad \text{5.7iii} \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box(s\in t), \bigwedge z\,[\Box(z\in t) \to z^{\mathfrak{A}}\in t/s^{\mathfrak{A}}] \Rightarrow s^{\mathfrak{A}}\in t \\ \hline
s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box(s\in t), [I\in t/s^{\mathfrak{A}}] \wedge \bigwedge z\,[\Box(z\in t) \to z^{\mathfrak{A}}\in t/s^{\mathfrak{A}}] \Rightarrow s^{\mathfrak{A}}\in t
\end{array} .
$$

Re 5.19vii. Employ 5.19ii, 5.19v, and 5.19vi. The inference marked by $*$ is according to 135.20vii in [**15**]:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow s^{\mathfrak{A}}\in\mathbf{Z}_2}{\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\Rightarrow\square(s^{\mathfrak{A}}\in\mathbf{Z}_2)}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2,\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\Rightarrow\mathfrak{P}\mathrm{i}_2[s,b^{\mathfrak{A}}]\qquad\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2,\square(s\in b),\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\Rightarrow s^{\mathfrak{A}}\in b}{s\in\breve{\mathbf{\Pi}}^{\circ}_2,s\in b^{\mathfrak{A}},\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\Rightarrow s^{\mathfrak{A}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2},\mathfrak{P}\mathrm{i}_2[s,b^{\mathfrak{A}}]\supset s\in b^{\mathfrak{A}}\Rightarrow\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\supset s^{\mathfrak{A}}\in b}\,*}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2},\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s,y]\supset s\in y)\Rightarrow\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\supset s^{\mathfrak{A}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2},\square(s^{\mathfrak{A}}\in\mathbf{Z}_2),\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s,y]\supset s\in y)\Rightarrow\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\supset s^{\mathfrak{A}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2},\square(s^{\mathfrak{A}}\in\mathbf{Z}_2)\mathbin{\scriptstyle\square}\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s,y]\supset s\in y)\Rightarrow\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\supset s^{\mathfrak{A}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 3}\Rightarrow\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},b]\supset s^{\mathfrak{A}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 3}\Rightarrow\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},y]\supset s^{\mathfrak{A}}\in y)}}{\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\Rightarrow\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},y]\supset s^{\mathfrak{A}}\in y)}}{[\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2}\Rightarrow\square(s^{\mathfrak{A}}\in\mathbf{Z}_2)\mathbin{\scriptstyle\square}\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{A}},y]\supset s^{\mathfrak{A}}\in y)}}{}}{}}{}}{[\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2}\Rightarrow s^{\mathfrak{A}}\in\lambda x\,(\square(x\in\mathbf{Z}_2)\mathbin{\scriptstyle\square}\bigwedge y\,(\mathfrak{P}\mathrm{i}_2[x,y]\supset x\in y))}\,.$$

Re 5.19viii. Employ 5.19i and 5.19vii. Completely straightforward, but nevertheless, here is a deduction:

$$\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_2}{\Rightarrow\square(I\in\breve{\mathbf{\Pi}}^{\circ}_2)}\qquad\dfrac{[\square(a\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2}\Rightarrow a^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2}{\square\square(a\in\breve{\mathbf{\Pi}}^{\circ}_2)\Rightarrow\square(a^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2)}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)}\,5.17\mathrm{i}\,.$$

Re 5.19ix. This is an immediate consequence of 5.19viii.
Re 5.19x. This is a straightforward consequence of 5.19viii, ix, and vii:

$$\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_2\mathbin{\scriptstyle\square}s\in\breve{\mathbf{\Pi}}^{\circ}_2\qquad\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\quad s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)}{s\in\breve{\mathbf{\Pi}}^{\circ}_2\mathbin{\scriptstyle\square}s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\mathbin{\scriptstyle\square}\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)}\quad\dfrac{\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2),\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\Rightarrow s^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2}{\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\mathbin{\scriptstyle\square}\square(s\in\breve{\mathbf{\Pi}}^{\circ}_2)\Rightarrow s^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2\mathbin{\scriptstyle\square}s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow s^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2}\,\clubsuit}{s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow s^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2}\,\clubsuit\,.$$

QED

REMARK 5.20. Notice the strange detour in the deduction of $s\in\breve{\mathbf{\Pi}}^{\circ}_2\Rightarrow s^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2$. I should very much like to call it a detour through infinity.

COROLLARY 5.21. *Inferences according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-derivable.*

(5.21i) $$\dfrac{\Gamma,s^{\mathfrak{A}}\in\breve{\mathbf{\Pi}}^{\circ}_2,\Pi\Rightarrow C}{\Gamma,s\in\breve{\mathbf{\Pi}}^{\circ}_2,\Pi\Rightarrow C}$$

(5.21ii)
$$\frac{\Gamma, s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \Pi, s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \Theta \Rightarrow C}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \Gamma, \Pi, \Theta \Rightarrow C}$$

As in the case of $\breve{\mathbf{\Pi}}^{\circ}$, this provides for a form of “induction”.

THEOREM 5.22. *Inferences according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

(5.22i)
$$\frac{\mathfrak{F}[I] \Rightarrow \qquad \mathfrak{F}[a^{2I}], \Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2}) \Rightarrow \Box\mathfrak{F}[a]}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \mathfrak{F}[s] \Rightarrow C}$$

(5.22ii)
$$\frac{\Rightarrow \mathfrak{F}[I] \qquad \Box\mathfrak{F}[a], \Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2}) \Rightarrow \mathfrak{F}[a^{2I}]}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[s]}$$

(5.22iii)
$$\frac{\Rightarrow \mathfrak{F}[I] \qquad [\Box\mathfrak{F}[a]]^{\cdot\mathrm{n}}, \Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2}) \Rightarrow \mathfrak{F}[a^{2I}]}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \Box\mathfrak{F}[s]}$$

Proof. Straightforward consequences of 5.17i and 5.21 in the usual way. *Re* 5.22ii. Employ 5.19x:

$$\dfrac{\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_{2} \qquad \Rightarrow \mathfrak{F}[I]}{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[I]}}{\Rightarrow \Box(I\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[I])} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_{2}}{a\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \mathfrak{F}[a] \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_{2}}}{a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a] \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_{2}}}{\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a]) \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_{2}} \qquad \dfrac{\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2}), \Box\,\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{2I}]}{[\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a])]^{\cdot 2} \Rightarrow \mathfrak{F}[a^{2I}]}}{[\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a])]^{\cdot 3} \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a^{2I}]}}{\Box\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a]) \Rightarrow \Box(a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[a^{2I}])}}{\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[s])}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \mathbin{\scriptstyle\square} \mathfrak{F}[s]}}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[s]}}$$

Re 5.22iii.

$$
\dfrac{\dfrac{\Rightarrow \mathfrak{F}[I]}{\Rightarrow \Box\mathfrak{F}[I]} \qquad \dfrac{\dfrac{[\Box\mathfrak{F}[a]]^{\cdot\mathrm{n}}, \Box(a \in \breve{\mathbf{\Pi}}^{\circ}_{2}) \Rightarrow \mathfrak{F}[a^{2I}]}{[\Box\mathfrak{F}[a]]^{\cdot\mathrm{n}}, a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[a^{2I}]}}{\Box\Box\mathfrak{F}[a], \Box(a \in \breve{\mathbf{\Pi}}^{\circ}_{2}) \Rightarrow \Box\mathfrak{F}[a^{2I}]}}{s \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \Box\mathfrak{F}[s]}
$$

QED

The next proposition somewhat corresponds to proposition 134.9 in [**15**], p. 1829.

PROPOSITION 5.23. *Sequents according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}_2}_{\lambda}$*-deducible.*

(5.23i) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A/s], [B/s] \Rightarrow [A \mathbin{\scriptstyle\Box} B/s]$

(5.23ii) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A \to B/s], [A/s] \Rightarrow [B/s]$

(5.23iii) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A \vee \neg A/s^{2I}], A \Rightarrow \Box A$

(5.23iv) $[A \vee \neg A/I], \Box A \Rightarrow [A/I]$

(5.23v) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A \vee \neg A/s^{2I}], \Box A \Rightarrow [A/s^{2I}]$

(5.23vi) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A \vee \neg A/s], \Box A \Rightarrow [A/s]$

Proof. *Re* 5.23i. In principle as for 5.23ii; left to the reader.
Re 5.23ii. I only show the "induction step":

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \to B/a] \Rightarrow [A \to B/a] \qquad [A/a] \Rightarrow [A/a]}{[A \to B/a], [A/a] \Rightarrow [A \to B/a] \mathbin{\scriptstyle\Box} [A/a]} \qquad [B/a] \Rightarrow [B/a]}{[A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a], [A \to B/a], [A/a] \Rightarrow [B/a]}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]), \Box[A \to B/a], \Box[A/a] \Rightarrow \Box[B/a]}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]), [A \to B/a^{2I}], [A/a^{2I}] \Rightarrow [B/a^{2I}]}\ 5.7\mathrm{i}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]), [A \to B/a^{2I}] \mathbin{\scriptstyle\Box} [A/a^{2I}] \Rightarrow [B/a^{2I}]}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]) \Rightarrow [A \to B/a^{2I}] \mathbin{\scriptstyle\Box} [A/a^{2I}] \to [B/a^{2I}]}\ .
$$

Re 5.23iii. Employ 134.18i from [**15**], p. 1833:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\square(A\vee\neg A), A \Rightarrow \square A}{[A\vee\neg A/I^{2I}], A \Rightarrow \square A}}{[A\vee\neg A/I^{2I}]\,{\scriptstyle\square}\,A \Rightarrow \square A}}{\Rightarrow [A\vee\neg A/I^{2I}]\,{\scriptstyle\square}\,A \to \square A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\vee\neg A/a^{2I}], A \Rightarrow [A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \qquad \square A \Rightarrow \square A}{[A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \to \square A, [A\vee\neg A/a^{2I}], A \Rightarrow \square A}}{[A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \to \square A, \square[A\vee\neg A/a^{2I}], A \Rightarrow \square A}}{[A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \to \square A, [A\vee\neg A/a^{2I2I}], A \Rightarrow \square A}}{[A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \to \square A, [A\vee\neg A/a^{2I2I}]\,{\scriptstyle\square}\,A \Rightarrow \square A}}{[A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \to \square A \Rightarrow [A\vee\neg A/a^{2I2I}]\,{\scriptstyle\square}\,A \to \square A}}{\square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,A \to \square A) \Rightarrow [A\vee\neg A/a^{2I2I}]\,{\scriptstyle\square}\,A \to \square A}}{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A\vee\neg A/s^{2I}]\,{\scriptstyle\square}\,A \to \square A}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/s^{2I}], A \Rightarrow \square A}}\ .$$

Re 5.23iv. Employ 134.13i from [**15**], p. 1831:

$$\dfrac{\dfrac{\square A \Rightarrow A}{\square A \Rightarrow [A/I]}}{[A\vee\neg A/I], \square A \Rightarrow [A/I]}\ .$$

Re 5.23v. I only show the "induction step"; employ 5.23iii:

$$\begin{array}{c}
[A\vee\neg A/a^{2I}] \Rightarrow [A\vee\neg A/a^{2I}] \qquad a\in\breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a^{2I}], A \Rightarrow \square A \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a^{2I}], [A\vee\neg A/a^{2I}], A \Rightarrow [A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A \\
\hline
\square a\in\breve{\mathbf{\Pi}}^{\circ}_2, \square[A\vee\neg A/a^{2I}], \square A \Rightarrow \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A) \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, \square[A\vee\neg A/a^{2I}], \square A \Rightarrow \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A) \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a^{2I2I}], \square A \Rightarrow \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A) \qquad \square[A/a^{2I}] \Rightarrow [A/a^{2I2I}] \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A) \to \square[A/a^{2I}], [A\vee\neg A/a^{2I2I}], \square A \Rightarrow [A/a^{2I}] \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A \to [A/a^{2I}]), [A\vee\neg A/a^{2I2I}]\,{\scriptstyle\square}\,\square A \Rightarrow [A/a^{2I}] \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A \to [A/a^{2I}]), [A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A \Rightarrow [A/a^{2I}] \\
\hline
a\in\breve{\mathbf{\Pi}}^{\circ}_2, \square([A\vee\neg A/a^{2I}]\,{\scriptstyle\square}\,\square A \to [A/a]) \Rightarrow [A\vee\neg A/a^{2I2I}]\,{\scriptstyle\square}\,\square A \to [A/a^{2I}]
\end{array}\ .$$

Re 5.23vi. Straightforward consequence of 5.23v and 5.23iv by 5.22ii.

QED

REMARK 5.24. Apparently, 134.9iv in [**15**], p. 1829, doesn't survive in the form

$$s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [A \vee \neg A/s], A \Rightarrow [A/s]\,.$$

In order to see this, confront

$$A \vee \neg A, A \vee \neg A, A \Rightarrow A \mathbin{\scriptstyle\square} A \mathbin{\scriptstyle\square} A$$

which is obviously $\mathbf{L^{i}D}_{\lambda}$-deducible, with

$$\Box\Box(A \vee \neg A), A \Rightarrow \Box\Box A$$

which, apparently, is not $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$-deducible. But while 134.9iv survives in some form, at least, *viz.*, as 5.23vi, there doesn't seem to be anything corresponding to 134.9ii, *i.e.*, something like

$$s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [A \mathbin{\scriptstyle\square} B/s] \Rightarrow [A/s] \mathbin{\scriptstyle\square} [B/s]$$

perhaps. This can be seen from the following consideration: while

$$(A \mathbin{\scriptstyle\square} B) \mathbin{\scriptstyle\square} (A \mathbin{\scriptstyle\square} B) \Rightarrow (A \mathbin{\scriptstyle\square} A) \mathbin{\scriptstyle\square} (B \mathbin{\scriptstyle\square} B)$$

is $\mathbf{L^{i}D}_{\lambda}$-deducible,

$$\Box(A \mathbin{\scriptstyle\square} B) \Rightarrow \Box A \mathbin{\scriptstyle\square} \Box B$$

is, apparently, not $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$-deducible.[16] This may be taken to indicate that $\boxed{\scriptstyle 2}$ is not just a repetition of the same kind of necessity operator that is already available in $\Box$.

5d. Applications. Just as $\breve{\mathbf{\Pi}}^{\circ}$ could be employed to define notions of necessity and weak implication, so can $\breve{\mathbf{\Pi}}^{\circ}_{2}$.

DEFINITION 5.25. $\boxed{\scriptstyle 2}A :\equiv \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x])\,.$

The following proposition corresponds in an obvious way to proposition 134.13 in [**15**], p. 1831.

PROPOSITION 5.26. *Sequents according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}_2}_{\lambda}$*-deducible.*

(5.26i) $\boxed{\scriptstyle 2}A \Rightarrow A$

(5.26ii) $\boxed{\scriptstyle 2}A \Rightarrow \Box A$

(5.26iii) $\boxed{\scriptstyle 2}A \Rightarrow \Box^{\mathrm{n}} A$

(5.26iv) $\boxed{\scriptstyle 2}A \Rightarrow \boxed{\scriptstyle 2}\Box A$

[16] Cf. remark 135.13 in [**15**], p. 1844.

(5.26v) $\boxed{2}\Box^{\mathrm{n}} A \Rightarrow \boxed{2}\Box^{\mathrm{n}'} A$

(5.26vi) $\boxed{2} A \Rightarrow \boxed{2}\Box^{\mathrm{n}} A$

(5.26vii) $\boxed{2} A, \boxed{2} B \Rightarrow \boxed{2}(A \mathbin{\scriptstyle\Box} B)$

(5.26viii) $\boxed{2}(A \rightarrow B), \boxed{2} A \Rightarrow \boxed{2} B$

(5.26ix) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \boxed{2}(s \in \breve{\mathbf{\Pi}}^{\circ}_{2})$

(5.26x) $s \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [s \in \breve{\mathbf{\Pi}}^{\circ}_{2}/s]$

Proof. Re 5.26i. As for 134.13i in [**15**], p. 1831, only with $\breve{\mathbf{\Pi}}^{\circ}_{2}$ instead of $\breve{\mathbf{\Pi}}^{\circ}$.
Re 5.26ii and 5.26iii. In view of 5.26iv, these are left to the reader.
Re 5.26iv.

$$\dfrac{\dfrac{\dfrac{\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_{2} \qquad \dfrac{\Box A \Rightarrow \Box A}{a \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A/a^{2I}] \Rightarrow [\Box A/a]}\;5.17\text{iii}}{a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/a^{2I}], a \in \breve{\mathbf{\Pi}}^{\circ}_{2}, a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [\Box A/a]}}{a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/a^{2I}], a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [\Box A/a]}\;5.21\text{ii}}{\dfrac{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x]), a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [\Box A/a]}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x]) \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [\Box A/a]}}}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x]) \Rightarrow \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [\Box A/x])}$$

Re 5.26v. This is just 5.26iv, only with $\Box^{\mathrm{n}} A$ being substituted for A.
Re 5.26vi. Employ an induction on n, based on 5.26v and 5.26vi.
Re 5.26vii. Employ 5.23i:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \qquad [A/a], [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [A \mathbin{\scriptstyle\Box} B/a]}{a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_{2}, a \in \breve{\mathbf{\Pi}}^{\circ}_{2}, a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [A \mathbin{\scriptstyle\Box} B/a]}}{a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [A \mathbin{\scriptstyle\Box} B/a]}\;5.21\text{ii}}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [B/x]), a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [A \mathbin{\scriptstyle\Box} B/a]}}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [B/x]) \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A \mathbin{\scriptstyle\Box} B/x]}}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [B/x]) \Rightarrow \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A \mathbin{\scriptstyle\Box} B/x])}$$

Re 5.26viii. Essentially, what has to be shown is that

$$\Rightarrow [A \rightarrow B/I] \mathbin{\scriptstyle\Box} [A/I] \rightarrow [B/I], \text{ and}$$

$$\Box([A \rightarrow B/a] \mathbin{\scriptstyle\Box} [A/a] \rightarrow [B/a]) \Rightarrow [A \rightarrow B/a^{2I}] \mathbin{\scriptstyle\Box} [A/a^{2I}] \rightarrow [B/a^{2I}]$$

are $\mathbf{L^iD}_\lambda^{Z_2}$-deducible. The first one is completely straightforward. As regards the second one, employ 5.23ii and proceed as for 5.26vii:

$$\begin{array}{c}
a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \qquad\qquad [A\to B/a],[A/a],a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow [B/a]\\
\hline\hline
a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A\to B/a], a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A/a], a\in\breve{\boldsymbol{\Pi}}^{\circ}_2, a\in\breve{\boldsymbol{\Pi}}^{\circ}_2, a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow [B/a]\\
\hline
a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A\to B/a], a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A/a], a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow [B/a]\\
\hline\hline
\bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A\to B/x]), \bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A/x]), a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow [B/a]\\
\hline
\bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A\to B/x]), \bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A/x]) \Rightarrow a\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [B/a]\\
\hline
\bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A\to B/x]), \bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [A/x]) \Rightarrow \bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [B/x])
\end{array}\quad 5.21\text{ii}$$

Re 5.26ix. Employ 5.19iv and 5.19x;

$$\begin{array}{c}
\dfrac{\Rightarrow I\in\breve{\boldsymbol{\Pi}}^{\circ}_2}{\Rightarrow \boxed{2}(I\in\breve{\boldsymbol{\Pi}}^{\circ}_2)} \qquad \dfrac{c\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow c^{2I}\in\breve{\boldsymbol{\Pi}}^{\circ}_2}{\boxed{2}(c\in\breve{\boldsymbol{\Pi}}^{\circ}_2) \Rightarrow \boxed{2}(c^{2I}\in\breve{\boldsymbol{\Pi}}^{\circ}_2)}\\
\hline
s\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow \boxed{2}(s\in\breve{\boldsymbol{\Pi}}^{\circ}_2)
\end{array}$$

Re 5.26x. Employ 5.26ix:

$$\begin{array}{c}
s\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow \bigwedge x\,(x\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [s\in\breve{\boldsymbol{\Pi}}^{\circ}_2/x])\\
\hline
s\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow s\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \to [s\in\breve{\boldsymbol{\Pi}}^{\circ}_2/s]\\
\hline
s\in\breve{\boldsymbol{\Pi}}^{\circ}_2, s\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow [s\in\breve{\boldsymbol{\Pi}}^{\circ}_2/s]\\
\hline
s\in\breve{\boldsymbol{\Pi}}^{\circ}_2 \Rightarrow [s\in\breve{\boldsymbol{\Pi}}^{\circ}_2/s]
\end{array}$$

QED

PROPOSITION 5.27. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda^{Z_2}$*-derivable.*

(5.27i) $$\frac{\Rightarrow A}{\Rightarrow \Box A}$$

(5.27ii) $$\frac{\Box^{\mathrm{n}}A, \Gamma \Rightarrow B}{\boxed{2}A, \Gamma \Rightarrow B}$$

(5.27iii) $$\frac{\Box A \Rightarrow B}{\boxed{2}A \Rightarrow \boxed{2}B}$$

(5.27iv) $$\frac{\Box^{\mathrm{n}}A \Rightarrow B}{\boxed{2}A \Rightarrow \boxed{2}B}$$

(5.27v) $$\dfrac{\Box^{\mathrm{n}}\Gamma \Rightarrow C}{\boxed{2}\Gamma \Rightarrow \boxed{2}C}$$

(5.27vi) $$\dfrac{\Gamma \Rightarrow A}{s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, \boxed{2}\Gamma \Rightarrow [A/s]}$$

Proof. *Re* 5.27i.

$$\dfrac{\dfrac{\dfrac{\Rightarrow [A/I] \qquad \dfrac{\Box[A/c] \Rightarrow \Box[A/c]}{\Box[A/c] \Rightarrow [A/c^{2I}]}}{a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [A/a]}}{\Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/a]}}{\Rightarrow \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x])}$$

Re 5.27ii–5.27vi. These are all fairly straightforward consequences of the results from 5.26 by means of 5.27i. I only show 5.27iv as an example. Employ 5.26vi and 5.26viii:

$$\dfrac{\boxed{2}A \Rightarrow \boxed{2}\Box^{\mathrm{n}}A \qquad \dfrac{\dfrac{\dfrac{\Box^{\mathrm{n}}A \Rightarrow B}{\Rightarrow \Box^{\mathrm{n}}A \rightarrow B}}{\Rightarrow \boxed{2}(\Box^{\mathrm{n}}A \rightarrow B)}\,5.27\mathrm{i} \qquad \boxed{2}(\Box^{\mathrm{n}}A \rightarrow B), \boxed{2}\Box^{\mathrm{n}}A \Rightarrow \boxed{2}B}{\boxed{2}\Box^{\mathrm{n}}A \Rightarrow \boxed{2}B}\,\clubsuit}{\boxed{2}A \Rightarrow \boxed{2}B}\,\clubsuit\,.$$

QED

REMARK 5.28. In view of 5.27ii and 5.27iv above, we can now say $\boxed{2}$ realizes the intention of $\boxed{\Box}$. The new symbol is chosen to allow a further development of the hierarchy: $\boxed{3}$, $\boxed{4}$, *etc.*, with $\boxed{1}$, of course, being $\Box$.

PROPOSITION 5.29. *Sequents according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.29i) $\boxed{2}(A \vee \neg A), \Box A \Rightarrow \boxed{2}A$

(5.29ii) $\boxed{2}(A \vee \neg A), \boxed{2}A \rightarrow B \Rightarrow \Box A \rightarrow B$

Proof. Re 5.29i. Employ 5.23vi:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \qquad a\in\breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a], \Box A \Rightarrow [A/a]}{a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/a], \Box A \Rightarrow [A/a]}}{a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/a], \Box A \Rightarrow [A/a]}\,5.21\text{ii}}{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/a], \Box A \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a]}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/x]), \Box A \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a]}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/x]), \Box A \Rightarrow \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x])}\ .$$

Re 5.29ii. As for 134.18ii in [**15**], p. 1833, this is a straightforward consequence of the foregoing result, in this case 5.29i:

$$\dfrac{\dfrac{\boxed{2}(A\vee\neg A), A \Rightarrow \boxed{2}A \qquad B \Rightarrow B}{\boxed{2}(A\vee\neg A), \boxed{2}A \rightarrow B, \Box A \Rightarrow B}}{\boxed{2}(A\vee\neg A), \boxed{2}A \rightarrow B \Rightarrow \Box A \rightarrow B}\ .$$

QED

With the notion of $\boxed{2}$ available, a form of induction with side-wffs, *i.e.*, induction under assumptions, can be established for $\breve{\mathbf{\Pi}}^{\circ}_2$-induction, just as in the case of $\breve{\mathbf{\Pi}}^{\circ}$ and $\Box$.

PROPOSITION 5.30. *Inferences according to the following schema are* $\mathbf{L}^{\mathrm{i}}\mathbf{D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

$$\dfrac{\Gamma \Rightarrow \mathfrak{F}[I] \qquad \Box\mathfrak{F}[a], a\in\breve{\mathbf{\Pi}}^{\circ}_2, \Gamma \Rightarrow \mathfrak{F}[a^{2I}]}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, \boxed{2}\Gamma \Rightarrow \mathfrak{F}[s]}$$

Proof. Let $\xi :\equiv \lambda x\,\mathfrak{F}[x]$. Employ 5.19x:

$$\dfrac{\Gamma \Rightarrow \mathfrak{F}[I]}{\Gamma \Rightarrow I \in \xi} \qquad \dfrac{\dfrac{\Box\, \mathfrak{F}[a], \Gamma, s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[a^{\mathit{2I}}]}{\Box\,(a \in \xi), \Gamma, s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a^{\mathit{2I}} \in \xi}}{\Gamma, s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)}$$

$$\dfrac{\Gamma, s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s], [s \in \breve{\mathbf{\Pi}}^{\circ}_2/s] \Rightarrow [\Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)/s]} \; 5.17\text{iv}$$

$$\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s], [s \in \breve{\mathbf{\Pi}}^{\circ}_2/s] \Rightarrow [\Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)/s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s], s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)/s]} \; 5.26\text{x}$$

$$\dfrac{\dfrac{\Gamma \Rightarrow I \in \xi}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow [I \in \xi/s]} \qquad \dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s], s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)/s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow [\Box\,(a \in \xi) \rightarrow (a^{\mathit{2I}} \in \xi)/s]} \; 5.21\text{ii}}{\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow \bigwedge z\, [\Box\,(z \in \xi) \rightarrow (z^{\mathit{2I}} \in \xi)/s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow \mathfrak{P}\mathrm{i}_2[s, \xi]}}$$

$$\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow \mathfrak{P}\mathrm{i}_2[s, \xi] \qquad s \in \xi \Rightarrow \mathfrak{F}[s]}{\Box(s \in \breve{\mathbf{\Pi}}^{\circ}_2), \Box[\Gamma/s], \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}$$

$$\dfrac{\Box(s \in \breve{\mathbf{\Pi}}^{\circ}_2), \Box[\Gamma/s], \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}{\Box(s \in \breve{\mathbf{\Pi}}^{\circ}_2), [\Gamma/s^{\mathit{2I}}], \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]} \; 5.7\text{i}$$

$$\dfrac{\Box(s \in \breve{\mathbf{\Pi}}^{\circ}_2), [\Gamma/s^{\mathit{2I}}], \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s^{\mathit{2I}}], \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]} \; 5.26\text{ix}$$

$$\dfrac{\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s^{\mathit{2I}}], \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s^{\mathit{2I}}], \Box(s \in \mathbf{Z}_2), \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}}{\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s^{\mathit{2I}}], \Box(s \in \mathbf{Z}_2) \mathbin{\square} \mathfrak{P}\mathrm{i}_2[s, \xi] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s^{\mathit{2I}}], s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}}$$

$$\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathit{2I}} \in \breve{\mathbf{\Pi}}^{\circ}_2 \qquad s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s^{\mathit{2I}}], s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}{\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, s \in \breve{\mathbf{\Pi}}^{\circ}_2, s^{\mathit{2I}} \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [\Gamma/s^{\mathit{2I}}], s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}{\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2, s \in \breve{\mathbf{\Pi}}^{\circ}_2, \boxed{2}\Gamma, s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, \boxed{2}\Gamma \Rightarrow \mathfrak{F}[s]}}}$$

QED

As in the case of $\breve{\mathbf{\Pi}}^{\circ}$, it is useful to introduce some form of an inclusive version of $\breve{\mathbf{\Pi}}^{\circ}_2$.

DEFINITION 5.31. $\breve{\mathbf{\Pi}}_2 :\equiv \lambda x \bigvee y\, (\boxed{2}(y = x) \mathbin{\square} y \in \breve{\mathbf{\Pi}}^{\circ}_2)$.

PROPOSITION 5.32. *Inferences according to the following schemata are* $\mathbf{L}^{\mathrm{i}}\mathbf{D}_{\lambda}^{\mathrm{Z}_2}$*-derivable.*

(5.32i)
$$\dfrac{A \Rightarrow B}{s \in \breve{\mathbf{\Pi}}_2, [A/s] \Rightarrow [B/s]}$$

(5.32ii)
$$\dfrac{\Box A \Rightarrow B}{s \in \breve{\mathbf{\Pi}}_2, \Box[A/s] \Rightarrow [B/s]}$$

(5.32iii) $$\frac{\mathit{\Gamma} \Rightarrow B}{s \in \breve{\mathbf{\Pi}}_2, [\mathit{\Gamma}/s] \Rightarrow [B/s]}$$

Proof. Essentially consequences of the corresponding proposition 5.17 for the exclusive case. I shall only show 5.32i as an example.
Re 5.32i. Employ 5.17ii:

$$\begin{array}{c}
b \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/b] \Rightarrow [B/b] \\ \hline\hline
b = s, b = s, b \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s] \\ \hline
\Box(b = s), b \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s] \\ \hline
\boxed{\scriptstyle 2}(b = s), b \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s] \\ \hline
\boxed{\scriptstyle 2}(b = s) \mathbin{\square} b \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s] \\ \hline
\bigvee y\, (\boxed{\scriptstyle 2}(y = s) \mathbin{\square} y \in \breve{\mathbf{\Pi}}^{\circ}_2), [A/s] \Rightarrow [B/s] \\ \hline
s \in \breve{\mathbf{\Pi}}_2, [A/s] \Rightarrow [B/s]
\end{array}$$

QED

As in the case of $\breve{\mathbf{\Pi}}$, this gives rise to a notion of "weak" implication.

DEFINITION 5.33. $A \supset_2 B :\equiv \bigvee x\, (x \in \breve{\mathbf{\Pi}}_2 \mathbin{\square} ([A/x] \rightarrow B))$.

PROPOSITION 5.34. *Inferences according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.34i) $$\frac{[A]^{\cdot \mathrm{n}}, \mathit{\Gamma} \Rightarrow B}{\mathit{\Gamma} \Rightarrow A \supset_2 B}$$

(5.34ii) $$\frac{\Box^{\mathrm{n}} A, \mathit{\Gamma} \Rightarrow B}{\mathit{\Gamma} \Rightarrow A \supset_2 B}$$

(5.34iii) $$\frac{\mathit{\Gamma} \Rightarrow A \qquad \mathit{\Pi} \Rightarrow A \supset_2 B}{\boxed{\scriptstyle 2}\mathit{\Gamma}, \mathit{\Pi} \Rightarrow B}$$

(5.34iv) $$\frac{\mathit{\Gamma} \Rightarrow A \qquad B, \mathit{\Pi} \Rightarrow C}{A \supset_2 B, \boxed{\scriptstyle 2}\mathit{\Gamma}, \mathit{\Pi} \Rightarrow C}$$

(5.34v) $$\frac{(A \rightarrow B) \Rightarrow (C_1 \rightarrow (\ldots \rightarrow (C_{\mathrm{n}} \rightarrow B) \ldots))}{(A \supset_2 B) \Rightarrow (C_1 \supset_2 (\ldots \supset_2 (C_{\mathrm{n}} \supset_2 B) \ldots))}$$

(5.34vi) $$\frac{A_2, \mathit{\Gamma} \Rightarrow A_1 \qquad B_1, \Box^{\mathrm{n}} A_2, \mathit{\Pi} \Rightarrow B_2}{A_1 \supset_2 B_1, \mathit{\Gamma}, \mathit{\Pi} \Rightarrow A_2 \supset_2 B_2}$$

(5.34vii)
$$\frac{B, \square^{n} A, \Gamma \Rightarrow C}{A \supset_2 B, \Gamma \Rightarrow A \supset_2 C}$$

Proof. In view of the similarity to the case of $\supset$ in [**15**], propositions 135.17, 135.20 and 135.22, I leave the proof to the reader. QED

REMARK 5.35. In view of remark 5.24 above, inferences according to the following schema
$$\frac{\boxed{2}A, \Gamma \Rightarrow C}{\boxed{2}(A \vee \neg A), A, \Gamma \Rightarrow \boxed{2}C}$$
cannot be expected to be generally $\mathbf{L^{i}D}_{\lambda}^{Z_2}$-derivable.

I now turn to the reason why I have gone to all the trouble with the notion of $\boxed{2}$: nested double induction.

PROPOSITION 5.36. *Inferences according to the following schema are* $\mathbf{L^{i}D}_{\lambda}^{Z_2}$*-derivable.*

$$\frac{\Gamma, b \in \mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[0,b] \qquad \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \Pi, a \in \mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[a',0] \qquad \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \mathfrak{F}[a',b], a \in \mathbf{N}^{\circ}, b \in \mathbf{N}^{\circ}, \Xi \Rightarrow \mathfrak{F}[a',b']}{s \in \mathbf{N}^{\circ}, t \in \mathbf{N}^{\circ}, \boxed{2}\Gamma, \square\boxed{2}\Pi, \square\boxed{2}\Xi \Rightarrow \mathfrak{F}[s,t]}$$

Proof. The inference marked by $*_1$ is somewhat (give or take some weakenings) according to 136.11ii, and that marked by $*_2$ is according to 136.11iii in [**15**], p. 1863.

$$\frac{\bigwedge^{\circ} y\, \mathfrak{F}[a,y], \Pi, a \in \mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[a',0] \qquad \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \mathfrak{F}[a',b], a \in \mathbf{N}^{\circ}, b \in \mathbf{N}^{\circ}, \Xi \Rightarrow \mathfrak{F}[a',b']}{\square \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \square\Pi, \square\Xi, c \in \mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[a',c]}\; *_1$$

$$\frac{\square \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \square\Pi, \square\Xi, c \in \mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[a',c]}{\dfrac{\square \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \square\Pi, \square\Xi \Rightarrow \bigwedge^{\circ} y\, \mathfrak{F}[a',y]}{\boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \boxed{2}\Pi, \boxed{2}\Xi \Rightarrow \boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[a',y]}}$$

$$\frac{\Gamma, b \in \mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[0,b]}{\dfrac{\Gamma \Rightarrow \bigwedge^{\circ} y\, \mathfrak{F}[0,y]}{\boxed{2}\Gamma \Rightarrow \boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[0,y]}}$$

$$\frac{\boxed{2}\Gamma \Rightarrow \boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[0,y] \qquad \boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[a,y], \boxed{2}\Pi, \boxed{2}\Xi \Rightarrow \boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[a',y]}{\dfrac{s \in \mathbf{N}^{\circ}, \boxed{2}\Gamma, \square\boxed{2}\Pi, \square\boxed{2}\Xi \Rightarrow \boxed{2} \bigwedge^{\circ} y\, \mathfrak{F}[s,y]}{\dfrac{s \in \mathbf{N}^{\circ}, \boxed{2}\Gamma, \square\boxed{2}\Pi, \square\boxed{2}\Xi \Rightarrow \bigwedge^{\circ} y\, \mathfrak{F}[s,y]}{s \in \mathbf{N}^{\circ}, t \in \mathbf{N}^{\circ}, \boxed{2}\Gamma, \square\boxed{2}\Pi, \square\boxed{2}\Xi \Rightarrow \mathfrak{F}[s,t]}}}\; *_2$$

QED

6. General Corrections[17]

Note. This list does not necessarily cover obvious typos or silly little grammatical mistakes and it is far from being complete. I you find a mistake, please drop me a note at `uwe.petersen@asfpg.de`, and I promise that you will get a mention in the next list.

p. 30, line 4: replace "f(z) = y" by "g(z) = y".
p. 45, first line: replace "additive numbers" by "principal numbers".
— line 16 (DEFINITION 4.26), before "*multiplicative principal number*" insert "(2) An ordinal number α is called a".

p. 65, line 8 from the bottom (HISTORICAL NOTE 8.8), replace "Dedekind [1887]" by "Dedekind [1888]".

p. 75, line 11 from the bottom, replace "$\mathrm{f_k(x)} = \phi(\mathrm{x,k})$ for all x" by "$\mathrm{f_k(x)} = \phi(\mathrm{x,k}) + 1$ for all x".

p. 131, line 6 from the bottom, replace "$\mathfrak{C}[B]$" by "$\neg\mathfrak{C}[B]$".
— line 5 from the bottom, replace "$\mathfrak{F}[B] \to \mathfrak{C}[B]$" by "$\neg(\mathfrak{F}[B] \to \mathfrak{C}[B])$".

p. 149, last line, replace "$\neg A$" in the inference rule ($\perp_\mathrm{C}$) by "$(\neg A)$".

p. 161, line 7, (16.45v), replace the lower sequent "$\varGamma \Rightarrow \neg(A \to B)$" by "$\varGamma \Rightarrow \varDelta, \neg(A \to B)$".

p. 182, line 15 from the bottom, delete "clause (ii) of definitiom 18.16.".
— line 17 from the bottom, delete "indexset(s)!of wffs!downward saturated".

p. 183, line 19 from the bottom, replace "$C_1 \to C_2$" by "$C_1 \wedge C_2$".

p. 189, line 18, replace

$$\text{“}\mathfrak{F}_1[\neg U \vee V, \neg(\neg U \vee \neg V)] \wedge \ldots \wedge \mathfrak{F}_\mathrm{k}[\neg U \vee V, \neg(\neg U \vee \neg V)]''$$

by

$$\text{“}\mathfrak{F}_1[\neg U \vee V, \neg(\neg U \vee V)] \wedge \ldots \wedge \mathfrak{F}_\mathrm{k}[\neg U \vee V, \neg(\neg U \vee V)]''\,.$$

p. 192, line 14, replace

$$\text{“}\varGamma[A] \Rightarrow \varDelta[A], \varDelta[A] \wedge \mathfrak{C}[A]'' \qquad \text{by} \qquad \text{“}\varGamma[A] \Rightarrow \varDelta[A], \mathfrak{F}[A] \wedge \mathfrak{C}[A]''.$$

[17] With special thanks to Valerie Kerruish who detected most of the mistakes.

— lines 15 and 20, replace "$A \leftrightarrow B, \Gamma[B] \Rightarrow \Delta[B], \Delta[B] \wedge \mathfrak{C}[B]''$" by "$A \leftrightarrow B, \Gamma[B] \Rightarrow \Delta[B], \Delta[B] \wedge \mathfrak{C}[B]''$".

p. 197, line 8 from the bottom, delete "*indexvariable(s)!sentence*".

p. 198, line 2 from the bottom, replace "$\bot$-inference" by "$\bot_{\mathrm{C}}$-inference".

p. 200, the bottom:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A, \Pi \Rightarrow B\end{matrix}}{\Pi \Rightarrow A \to B}}{\Gamma, \Pi \Rightarrow B}\ .$$

instead of:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Pi \Rightarrow B\end{matrix}}{\Pi \Rightarrow A \to B}}{\Gamma, \Pi \Rightarrow B}\ .$$

p. 213, second line in top proof figure, right branch, replace "$\Theta, \Gamma[\,] \Rightarrow \Delta[\,], A, \Xi$" by "$\Theta, B, \Pi[\,] \Rightarrow \Lambda[\,], \Xi$".

p. 214, second line in top proof figure, left branch, replace "$\Pi, \Gamma[\,] \Rightarrow \Delta[\,], A, K, A \wedge B$" by "$\Pi, \Gamma[\,] \Rightarrow \Delta[\,], A, \Lambda, A \wedge B$".

p. 217, line 1, replace "$\mathrm{max(l, m) + 1 + r}$" by "$\mathrm{max(l, m) + 1 + r + 1}$".

p. 241, line 12 from the bottom, condition "(vi)": read "$\mathbf{LK}_0^0$" instead of "$\mathbf{GK}_0^0$".

p. 242, last line, read "Only the second one" instead of "Only the fourth".

p. 247, line 6, replace "dropping axioms HA13 and HA15" by "replacing axiom HA13 by $\neg\neg\bot \to \bot$ and dropping axiom HA15 completely".

p. 249, proof figure "*Re* 24.7iv", first line: read "$A \Rightarrow A \vee \neg A$" instead of "$A \Rightarrow A \Rightarrow \neg A$".

p. 250, lines 3–6 from the bottom, "(24.11i)–(24.11iv)", read "acc." instead of "max".

p. 301, line 13 from the bottom: read "contradictions" instead of "contractions".

p. 306, line 5: read “marked with the sign ♠” instead of “marked with an exclamation sign”.
— line 7 proposition 27.7: read “DSL” instead of “CDL”.

p. 307, replace proof figure in the middle of the page:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A}{\neg A, A \Rightarrow}}{A \to \neg A, A, A \Rightarrow}}{A \to \neg A, A \square A \Rightarrow} \qquad \dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}}{A \to \neg A, A \to A \square A \Rightarrow \neg A}}{A \to A \square A \Rightarrow (A \to \neg A) \to \neg A}\ .$$

by:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A} \qquad \dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A}{\neg A, A \Rightarrow}}{A \to \neg A, A, A \Rightarrow}}{A \to \neg A, A \square A \Rightarrow}}{A \to \neg A, A \to A \square A \Rightarrow \neg A}}{A \to A \square A \Rightarrow (A \to \neg A) \to \neg A}\ .$$

p. 309, second line: add “logic” after “dialectical”.

p. 316, first line: cancel 27.35viii; already 27.35vi;
— second line: read “$(A \diamond \neg A) \leftrightarrow \top$” instead of “$(A \diamond \neg A) \leftrightarrow \bot$”.

p. 352, l. 10 from the bottom (Corollary 30.21): read “30.21i” instead of “30.80”;
— l. 11 from the bottom (Corollary 30.21): read “30.21ii” instead of “30.81”;
— l. 12 from the bottom: read “30.20i” instead of “30.17i”.

p. 460, after the first proof figure, replace: “A new deduction is being contracted as follows” by : “A new deduction can be constructed as follows”.

p. 466, l. 12 from the bottom, Definition 41.6: swap (1) and (2).

p. 468, last three lines: replace Definitions 41.14 by the following:

DEFINITIONS 41.14 (1) $uni[\mathfrak{F}] :\equiv \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \Box \mathfrak{F}[z_2] \to z_1 = z_2)$.
(2) $\iota x\, \mathfrak{F}[x] :\equiv \lambda x \bigwedge y\, (uni[\mathfrak{F}] \Box \mathfrak{F}[y] \to x \in y)$.

p. 469, replace PROPOSITION 41.15 by the following:
PROPOSITION 41.15. *Sequents according to the following schemata are* $\mathbf{LX}_\lambda^-$*-deducible for* $\mathbf{X} \in \{\mathbf{K}, \mathbf{J}, \mathbf{P}, \mathbf{D}\}$.

(41.15i) $s \in \iota x\, \mathfrak{F}[x], \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \Box \mathfrak{F}[z_2] \to z_1 = z_2), \mathfrak{F}[t] \Rightarrow s \in t$

(41.15ii) $s \in t, \mathfrak{F}[t] \Rightarrow s \in \iota x\, \mathfrak{F}[x]$

(41.15iii) $\bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \Box \mathfrak{F}[z_2] \to z_1 = z_2), \mathfrak{F}[t] \Rightarrow \iota x\, \mathfrak{F}[x] = t$

Proof. *Re* 41.15i.

$$\dfrac{\dfrac{\dfrac{uni[\mathfrak{F}], \mathfrak{F}[t] \Rightarrow uni[\mathfrak{F}] \Box \mathfrak{F}[t] \qquad s \in t \Rightarrow s \in t}{uni[\mathfrak{F}] \Box \mathfrak{F}[t] \to s \in t, uni[\mathfrak{F}], \mathfrak{F}[t] \Rightarrow s \in t}}{\bigwedge y\, (uni[\mathfrak{F}] \Box \mathfrak{F}[y] \to s \in y), uni[\mathfrak{F}], \mathfrak{F}[t] \Rightarrow s \in t}}{s \in \iota x\, \mathfrak{F}[x], \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \Box \mathfrak{F}[z_2] \to z_1 = z_2), \mathfrak{F}[t] \Rightarrow s \in t}\,.$$

Re 41.15ii.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{F}[t], \mathfrak{F}[b] \Rightarrow \mathfrak{F}[t] \Box \mathfrak{F}[b] \qquad b = t, s \in t \Rightarrow s \in b}{s \in t, \mathfrak{F}[t] \Box \mathfrak{F}[b] \to b = t, \mathfrak{F}[t], \mathfrak{F}[b] \Rightarrow s \in b}}{s \in t, \mathfrak{F}[t], \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \Box \mathfrak{F}[z_2] \to z_1 = z_2), \mathfrak{F}[b] \Rightarrow s \in b}}{s \in t, \mathfrak{F}[t], uni[\mathfrak{F}], \mathfrak{F}[b] \Rightarrow s \in b}}{s \in t, \mathfrak{F}[t], uni[\mathfrak{F}] \Box \mathfrak{F}[b] \Rightarrow s \in b}}{s \in t, \mathfrak{F}[t] \Rightarrow uni[\mathfrak{F}] \Box \mathfrak{F}[b] \to s \in b}}{s \in t, \mathfrak{F}[t] \Rightarrow \bigwedge y\, (uni[\mathfrak{F}] \Box \mathfrak{F}[y] \to s \in y)}}{s \in t, \mathfrak{F}[t] \Rightarrow s \in \iota x\, \mathfrak{F}[x]}\,.$$

Re 41.15iii. This is a fairly straightforward combination of 41.15i and ii. Left to the reader. QED

p. 484, l. 9 from the bottom "41.57iv": read "$t \in \mathbf{T}, s' = t', r \in s \Rightarrow r \in t$" instead of "$t \in \mathbf{T}, s' = t', s \in r \Rightarrow s \in t$".

p. 491, "*Re* 41.72iii": involves a cut which doesn't make it suitable for $\mathbf{LP}_\lambda$.

p. 492, proof figure "*Re* 41.74ii", replace "$\langle \langle s, 0' \rangle, \mathfrak{g}(s, 0, \mathfrak{f}(s)) \rangle \in \mathfrak{h}$" by

"$\langle\langle s, 0\rangle, \mathfrak{f}(s))\rangle \in \mathfrak{h}$"

p. 495, proof figure "*Re* 41.78i", second line: add "$t \in \mathbf{N}, \langle\langle s, t\rangle, r\rangle$" before "$\in \mathfrak{h}$,"
— proof figure "*Re* 41.79i", replace lower sequent by

$$s \in \mathbf{N} \Rightarrow \langle\langle s, 0\rangle, \mathfrak{h}(s, 0)\rangle \in \mathfrak{h}\,.$$

p. 499, l. 10 from the bottom (in REMARK 42.3): replace

$$\lambda x \centerdot t \equiv \{z : \bigvee x \bigvee y\, (z = \langle x, y\rangle) \,\Box\, y = t\}\,,$$

by

$$\lambda x \centerdot t \equiv \{z : \bigvee x \bigvee y\, (z = \langle x, y\rangle \,\Box\, y = t)\}\,,$$

— l. 3 from the bottom: replace **D** by **P**.

p. 502, l. 11 (DEFINITION 42.11. clause (6)) add: "with z being the first variable $\notin FV(AB)$".

p. 564, l. 6 from the bottom: "obtained from them" instead of "obtained form them".

p. 570, l. 6: read "$\mathbf{LX}_1^{\mathrm{Q}}$-admissible" instead of "$\mathbf{HX}_1^{\mathrm{Q}}$-admissible".

p. 572, l. 2: "the formal principles" instead of "the formal principle"
— l. 4: "The remainder of this section" instead of "The remainder this section"

p. 574, l. 14 and 15: "$\bigwedge x\, \mathfrak{F}[x]$" instead of "$\bigwedge x\, [x]$"

p. 586, l. 4 (proof figure, second line): read "$b < c, c < a', b < a \to \neg\mathfrak{F}[b]$" instead of "$\neg\mathfrak{F}[b]$)"
— last line: replace

$$\neg\neg \bigvee x\, \mathfrak{F}[x] \Rightarrow \bigvee y\, (\mathfrak{F}[y] \wedge \bigwedge z\, (z < y \to \neg\mathfrak{F}[z]))\,.$$

by

$$\neg\neg(\bigvee x\, \mathfrak{F}[x] \to \bigvee y\, (\mathfrak{F}[y] \wedge \bigwedge z\, (z < y \to \neg\mathfrak{F}[z])))\,.$$

p. 605, l. 18 from the bottom (DEFINITION 48.4 (2)): replace "*fof* (r, s)" by "*fof* (r)".

p. 607, l. 6 from the bottom (main text): replace "free variables.footnote" by "free variables." and read the sentence beginning with "Primitive recursive functions" and ending with "variables in PRA." as a footnote.

p. 621, l. 4 (REMARK 48.61): read

$$\bigwedge x\,(\mathfrak{F}[x] \vee \neg\mathfrak{F}[x]) \Rightarrow \bigvee e \bigwedge x\,((\mathfrak{F}[x] \leftrightarrow \phi_e(x)=1) \wedge (\phi_e(x)=0 \vee \phi_e(x)=1)),$$

instead of

$$\bigwedge x\,(\mathfrak{F}[x] \vee \mathfrak{F}[x]) \Rightarrow \bigvee e \bigwedge x\,((\mathfrak{F} \leftrightarrow \phi_e(x) = 1) \wedge (\phi_e(x) = 0 \vee \phi_e(x) = 1)).$$

p. 739, l. 19 (QUOTATION 57.27): read "in very few cases or none" instead of "in very few cases or non".

p. 894, l. 4 from the bottom (footnote 2): replace "Wang [1986]" by "Wang [1987]".

p. 1011, l. 8: read "from" instead of "form".

p. 1017, l. 14 from the bottom: read "it follows" instead of "if follows".

p. 1026, l. 13 from the bottom: read "This sounds like" instead of "This sound like".

p. 1030, l. 16: read "from the value" instead of "form the value".

p. 1081, l. 6, QUOTATION 76.16. (1), add: "Weyl" before "[1921]".

p. 1087, l. 23 from the bottom, QUOTATION 77.8. (1), new paragraph after "meaningless." and before "In all contexts ...".

p. 1087, l. 14 from the bottom, QUOTATION 77.8. (1), add: "put" before "numerals for the variables in such a way ...".

p. 1088, l. 22, QUOTATION 77.9. replace "ist" by "is": "correlate of a subclass is that subclass itself".

p. 109, after l. 24 (QUOTATION 78.12) add line: "expresses a true proposition with respect to every one of *these* models, we"

p. 1099, l. 11 from the bottom: replace "67.20" by "78.17".

p. 1102, l. 13: replace "Wang [1986]" by "Wang [1987]".

p. 1104, l. 4 from the bottom: replace "Wang [1986]" by "Wang [1987]".

p. 1106, l. 3: replace "Wang [1986]" by "Wang [1987]".

p. 1108, l. 16: replace "form" by "from" in QUOTATION 79.10.

p. 1109, l. 1: replace "Wang [1986]" by "Wang [1987]".

p. 1126, l. 19: replace "Wang [1986]" by "Wang [1987]".

p. 1133, l. 4: replace "*Brouwerian*" by "*Brouwerians*".

p. 1158, l. 3 from the bottom: replace "Wang [1986]" by "Wang [1987]".

p. 1161, l. 16: replace "Wang [1986]" by "Wang [1987]".

p. 1169, l. 4: read "share no fixed point" instead of "share not fixed point".

p. 1185, l. 16 from the bottom (QUOTATION 85.10 (4)): read "$\forall x_1 \dots \forall x_n t(x_1, \dots, x_n = \overline{o})$" instead of "$\forall x_1 \dots \forall x_n (t = \overline{o})$".

p. 1207, l. 5: replace "Wang [1986]" by "Wang [1987]".

p. 1300, l. 1: read "how many angels" instead of "how man angels".

p. 1303, l. 2: read "*d'être* for the" instead of "*d'être* n for the".
— l. 14 from the bottom: read "Gödel" instead of "G"odel".

p. 1307, l. 1: read "amenable" instead of "amendable".

p. 1368, l. 7 from the bottom, "(3) Girard [1995]": replace "p. 28" by "p. 171";
— l. 4 from the bottom: read "work" instead of "word";
— last line (of text): read "[1982]" instead of "[1974]".

p. 1387, l. 8 from the bottom (footnote 11): replace "Wang [1986]" by "Wang [1987]".

p. 1412, l. 4: replace "and Wandschneider [1984]" by "Kesselring [1984], and Wandschneider [1991]".

p. 1546, l. 13 from the bottom: replace "ἀριθμίζει" by "ἀριθμητίζει".

p. 1421, l. 9 from the bottom (disregarding footnotes): replace "ancient means "never"." by "ancient means "ever", "once"."

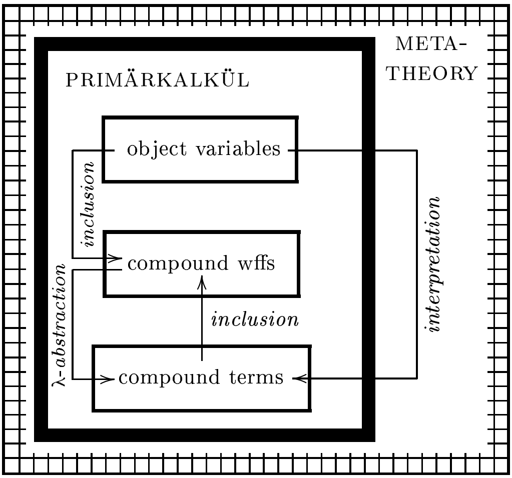
p. 1557, l. 6 from the bottom: delete "Take, *e.g.*, *tertium non datur* for negated wffs, $\neg A \vee \neg\neg A$; this is perfectly provable in intuitionistic logic". This is utter nonsense and I have no idea what was going on in my mind when I wrote it. Perhaps I was thinking of 'double negation', $\neg\neg A \to A$, which holds intuitionistically for negated wffs: $\neg\neg\neg A \to \neg A$. This is what it can be replaced by: "Take, *e.g.*, the double negation of *tertium non datur*, $\neg\neg(A \vee \neg A)$; this is perfectly provable in intuitionistic logic".

p. 1571, l. 2, replace the topmost proof figure by the following one:

$$\dfrac{\dfrac{R\in R \Rightarrow R\in R}{\Rightarrow R\in R, R\notin R} \qquad \dfrac{\dfrac{\dfrac{R\in R \Rightarrow R\in R}{R\notin R, R\in R \Rightarrow}}{R\in R, R\in R \Rightarrow}\,\in}{R\in R \Rightarrow}\,\spadesuit}{\Rightarrow R\notin R}\,\clubsuit$$

and (on the same page), in second proof figure, last line, read "$R\notin R \Rightarrow$" instead of "$R\in R \Rightarrow$".

p. 1601: replace diagram 116.10 by the following one:



p. 1621, l. 9 from the bottom: replace "$\bigwedge y$" by "$\bigwedge Y$".
p. 1630, l. 18: insert ")" before "$\Rightarrow$"; *i.e.*, replace "$(\mathfrak{F}[t_2] \vee \neg\mathfrak{F}[t_2] \Rightarrow$" by "$(\mathfrak{F}[t_2] \vee \neg\mathfrak{F}[t_2]) \Rightarrow$".
p. 1669: in the proof figure "*Re* 123.13ii", sixth line: replace "$\lambda\top \sqsubseteq \bigwedge x\,(\mathfrak{F}[x] \to \widetilde{b})$" by $\lambda\top \sqsubseteq \lambda \bigwedge x\,(\mathfrak{F}[x] \to \widetilde{b})$".
In remark 123.14, second line, replace "that" by "than".

p. 1670: in the proof figure "*Re* 123.18i" replace "$\bot$" by "$\top$" throughout. In the proof figure "*Re* 123.18ii" replace the second line "$A \to \widetilde{a} \Rightarrow B \to \widetilde{a}$" by "$B \to \widetilde{a} \Rightarrow A \to \widetilde{a}$" and the third $C \to (A \to \widetilde{a}) \Rightarrow C \to (B \to \widetilde{a})$" by "$C \to (B \to \widetilde{a}) \Rightarrow C \to (A \to \widetilde{a})$".

In the proof figure "*Re* 123.18iii" replace the second line "$A \to (C \to \widetilde{a}) \Rightarrow B \to (C \to \widetilde{a})$" by "$B \to (C \to \widetilde{a}) \Rightarrow A \to (C \to \widetilde{a})$".

p. 1705, (125.12iii): replace

$$s \ddot{=} t \Rightarrow \lambda x\,(\lambda y\,(y \ddot{=} x) \ddot{\subseteq} s) \ddot{\subseteq} \lambda x\,(\lambda y\,(y \ddot{=} x) \ddot{\subseteq} t))$$

by

$$s \ddot{\subseteq} t \Rightarrow \lambda x\,(\lambda y\,(y \ddot{=} x) \ddot{\subseteq} s) \sqsubseteq \lambda x\,(\lambda y\,(y \ddot{=} x) \ddot{\subseteq} t))$$

— *Re* 125.12iii: Replace proof figure by:

$$\frac{s \ddot{\subseteq} t, \lambda y\,(y \ddot{=} a) \ddot{\subseteq} s \Rightarrow \lambda y\,(y \ddot{=} a) \ddot{\subseteq} t}{s \ddot{\subseteq} t \Rightarrow \lambda x\,(\lambda y\,(y \ddot{=} x) \ddot{\subseteq} s) \sqsubseteq \lambda x\,(\lambda y\,(y \ddot{=} x) \ddot{\subseteq} t)}\ .$$

p. 1706, l. 2, proof of proposition 125.15, replace the whole thing by the following:

Proof. By means of contraction it is possible to prove a contradiction along similar lines as for 141.11 in the appendix A1, only with somewhat more rudimentary notions due to the lack of 41.2 (1.2) and (2.3). Take R to be defined as $\lambda x\,(\ddot{\{}x\ddot{\}} \ddot{\not\subseteq} x)$ and consider the following deductions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \qquad \dfrac{\Rightarrow a \ddot{=} a \qquad \dfrac{\ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\subseteq} a}{\ddot{\{}a\ddot{\}} \ddot{\not\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{\ddot{\{}a\ddot{\}} \ddot{\subseteq} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{\lambda x\,(\ddot{\{}x\ddot{\}} \ddot{\subseteq} a) \sqsubseteq \lambda x\,(\ddot{\{}x\ddot{\}} \ddot{\subseteq} \mathrm{R}), \ddot{\{}a\ddot{\}} \ddot{\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}\ 125.14}{a \ddot{\subseteq} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{\dfrac{a \ddot{=} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}{\dfrac{a \ddot{=} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}{a \ddot{=} \mathrm{R} \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\not\subseteq} a}}\ \spadesuit}}{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \ddot{\subseteq} \lambda x\,(\ddot{\{}x\ddot{\}} \ddot{\not\subseteq} x)}\ 125.7\text{i}\,,$$

and also:

$$\frac{a \ddot{=} \mathrm{R} \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\not\subseteq} a}{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \sqsubseteq \lambda x\,(\ddot{\{}x\ddot{\}} \ddot{\not\subseteq} x)}\ .$$

Continue as follows:

$$\dfrac{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \ddot{=} \ddot{\{}\mathrm{R}\ddot{\}} \qquad \dfrac{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \ddot{\subseteq} \mathrm{R}}{\ddot{\{}\mathrm{R}\ddot{\}} \ddot{\not\subseteq} \mathrm{R} \Rightarrow}}{\ddot{\{}\mathrm{R}\ddot{\}} \sqsubseteq \mathrm{R} \Rightarrow} \;.$$

QED

p. 1729, l. 4 from the bottom: insert round bracket after $\|^{\mathrm{DL}}$: $\neg\neg(s \in \|\lambda x\, \mathfrak{C}[x]\|^{\mathrm{DL}}) \Rightarrow s \in \|\lambda x\, \mathfrak{C}[x]\|^{\mathrm{DL}}$.

p. 1737, first line: replace “Employ 126.64ii” by “Employ 126.63i”.

p. 1759, l. 5, (128.27ii), replace h by f: $r_1 = s_1, r_2 = s_2, f[\![r_1, r_2]\!] = t \Rightarrow f[\![s_1, s_2]\!] = t$.

p. 1763, l. 15 from the bottom “128.34ii”: read “$\Rightarrow\ 0 \in \mathbf{T}$” instead of “$t \in \mathbf{T}, s \in r, r \in t \Rightarrow s \in t$”.

p. 1809, l. 16 from the bottom (proof of lemma 132.13, third last line): read “By proposition 126.35” instead of “By proposition 131.22”.

p. 1818, l. 7 from the bottom (PROPOSITION 133.8): read “$\mathcal{D}$ has the left rank 1 ” instead of “$\mathcal{D}$ has the rank 1”.
— l. 5 from the bottom: read “If the left rank were” instead of “If the rank were”.

p. 1823, l. 6 from the bottom, replace

$$\mathit{\Gamma}, \breve{\gamma}[\textstyle\bigwedge z\,(z \in b \to z^I \in b)] \Rightarrow s \in b$$

by

$$\mathit{\Gamma}, \breve{\gamma}[\textstyle\bigwedge z\,(z \in b \to z^I \in b)] \Rightarrow s \in s\,.$$

— Second last line, replace

$$\mathit{\Gamma}, [\textstyle\bigwedge z\,(z \in b \to z^I \in b)]^{\cdot \mathrm{n}} \Rightarrow s \in b$$

by

$$\mathit{\Gamma}, [\textstyle\bigwedge z\,(z \in b \to z^I \in b)]^{\cdot \mathrm{n}} \Rightarrow s \in c\,.$$

p. 1824, second line in the proof of theorem 135.15, replace “$\delta_1 \geq 0$” by “$\delta_1 > 0$”.

p. 1825, DEFINITION 134.1., replace

$$\breve{\mathbf{\Pi}}^{\circ} :\equiv \lambda x\,(x \in \mathbf{Z} \,\square \textstyle\bigwedge y\,(I \in y \,\square\, [\bigwedge z\,(z \in y \to z^I \in y)/x] \to x \in y))$$

by

$$\breve{\mathbf{\Pi}}^{\circ} :\equiv \lambda x\,(x\in\mathbf{Z}\,\square\,\bigwedge y\,([I\in y\wedge\bigwedge z\,(z\in y\to z^{I}\in y)/x]\to x\in y))\,.$$

p. 1830, first line (134.9iv), replace “131.18ii” by “131.18i”.

p. 1832, second last line (134.16iii), replace

$$\mathbf{L^iD}^{Z}_{\lambda}\cup\{\Box(\Box\bot\to\bot)\to\Box\bot\}\vdash\bot\Box(\Box\bot\to\bot)\to\Box\bot$$

by

$$\mathbf{L^iD}^{Z}_{\lambda}\cup\{\Box(\Box\bot\to\bot)\to\Box\bot\}\vdash\bot\,.$$

p. 1834, replace last proof figure on that page (“*Re* 134.22i.”) by the following one:

$$\dfrac{\Box\Gamma\Rightarrow\Box C \qquad \dfrac{\dfrac{\dfrac{\genfrac{}{}{1.5pt}{}{\Gamma\Rightarrow A}{C\Rightarrow A}}{\Rightarrow C\to A}}{\Rightarrow\Box(C\to A)} \qquad \Box(C\to A),\Box C\Rightarrow\Box A}{\Box C\Rightarrow\Box A}\,\clubsuit}{\Box\Gamma\Rightarrow\Box A}\,\clubsuit\,.$$

p. 1842, third line, replace

$$[B/s],[A/I\sqcap I]\to[B/I],[A/s],[A/I]\Rightarrow[B/s\sqcap I]$$

by

$$[B/s],[A/I\sqcap I]\to[B/I],[A/I],[A/I]\Rightarrow[B/s\sqcap I]$$

p. 1848, l. 12 from the bottom, first line in the deduction *re* 135.20vii, replace “$A_2\Rightarrow A_1$” by “$A_2,\Gamma\Rightarrow A_1$” and cancel “$B_1,A_2,A_2,\Gamma,\Pi\Rightarrow B_2$” completley.

— l. 11 from the bottom, second line of that deduction, replace

“$a\in\breve{\mathbf{\Pi}},[A_2/a]\Rightarrow[A_1/a]$” by “$a\in\breve{\mathbf{\Pi}},[A_2/a],\Gamma\Rightarrow[A_1/a]$”,

and

“$B_1,A_2,A_2,\Gamma,\Pi\Rightarrow B_2$” by “$B_1,A_2,A_2,\Pi\Rightarrow B_2$”.

p. 1886, third line (137.8i), as well as line 10 and 11 (in proof figure), replace “$\Rightarrow \mathit{zhf}[s_1,t_1,r_1]$” by “$\Rightarrow \mathit{zhf}[s_2,t_2,r_2]$”;

— fourth line (137.8ii), replace “$\Rightarrow \mathit{shg}[s_1,t_1,r_1]$” by “$\Rightarrow \mathit{shg}[s_2,t_2,r_2]$”;

p. 1901, sixth line from the bottom, replace “$\to y\in T$” by “$\to y\in\mathcal{T}$”.

— l. 3 from the bottom, replace "$\rightarrow \mathcal{T}$" by "$\rightarrow y\in\mathcal{T}$".

p. 1903, first line, replace the quantifier $\bigvee$ in the wff

$$\neg\bigvee x\,(ded_{\mathbf{L^iD}^{\mathrm{z}}_{\lambda}}(x,\ulcorner \mathrm{G}\urcorner)=0)\leftrightarrow \mathrm{G}$$

by the quantifier $\bigvee^{\circ}$:

$$\neg\bigvee^{\circ} x\,(ded_{\mathbf{L^iD}^{\mathrm{z}}_{\lambda}}(x,\ulcorner \mathrm{G}\urcorner)=0)\leftrightarrow \mathrm{G}\,.$$

p. 1923, second proof figure, replace

$$\dfrac{\Rightarrow \mathrm{C}_A\in\mathrm{C}_A \qquad \dfrac{\dfrac{\mathrm{C}_A\in\mathrm{C}_A \Rightarrow A}{\Rightarrow \mathrm{C}_A\in\mathrm{C}_A\rightarrow A}}{\mathrm{C}_A\in\mathrm{C}_A\Rightarrow A}}{\Rightarrow A}\,\clubsuit\,.$$

by

$$\dfrac{\dfrac{\dfrac{\mathrm{C}_A\in\mathrm{C}_A\Rightarrow A}{\Rightarrow \mathrm{C}_A\in\mathrm{C}_A\rightarrow A}}{\Rightarrow \mathrm{C}_A\in\mathrm{C}_A}\qquad \mathrm{C}_A\in\mathrm{C}_A\Rightarrow A}{\Rightarrow A}\,\clubsuit\,.$$

p. 1925, *Re* 141.iii and iv, third proof figure from the top, replace "$\Leftrightarrow \mathrm{K}\in\lambda x\,(\mathsf{C}x\notin x)$" by "$\Leftrightarrow \mathrm{K}\in\lambda x\,(\mathsf{C}x\in x)$"

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